

# HAUSDORFF DIMENSION OF THE GRAPHS OF THE CLASSICAL WEIERSTRASS FUNCTIONS

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**ABSTRACT.** We show that the graph of the classical Weierstrass function  $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$  has Hausdorff dimension  $2 + \log \lambda / \log b$ , for every integer  $b \geq 2$  and every  $\lambda \in (1/b, 1)$ . Replacing  $\cos(2\pi x)$  by a general non-constant  $C^2$  periodic function, we obtain the same result under a further assumption that  $\lambda b$  is close to 1.

## 1. INTRODUCTION

In this paper, we study the Hausdorff dimension of the graph of the following Weierstrass function

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x), x \in \mathbb{R}$$

where  $0 < \lambda < 1 < b$  and  $b\lambda > 1$ . These functions, studied by Weierstrass and Hardy [6], are probably the most well-known examples of continuous but nowhere differentiable functions. Study of the graph of these and related functions from a geometric point of view as fractal sets have attracted much attention since Besicovitch and Ursell [3]. A long standing conjecture asserts that the Hausdorff dimension of the graph of  $W_{\lambda,b}$  is equal to

$$D = 2 + \frac{\log \lambda}{\log b},$$

see for example [12]. Although the box dimension and packing dimension have been shown to be equal to  $D$  for a large class of functions including all the functions  $W_{\lambda,b}$  (see [7, 8, 14]), the conjecture about Hausdorff dimension remains open even in the case when  $b$  is an integer.

**Main Theorem.** *For any integer  $b \geq 2$  and any  $\lambda \in (b^{-1}, 1)$ , the Hausdorff dimension of the graph of the Weierstrass function  $W_{\lambda,b}$  is equal to  $D$ .*

More generally, we consider the following function:

$$f_{\lambda,b}^{\phi}(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x),$$

where  $\phi$  is a  $\mathbb{Z}$ -periodic function and  $\lambda, b$  are as above. So  $W_{\lambda,b}$  corresponds to the case  $\phi(x) = \cos(2\pi x)$ . Our method also shows the following:

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**Theorem 1.1.** *For any  $\mathbb{Z}$ -periodic, non-constant  $C^2$  function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and any integer  $b \geq 2$  there exists  $K_0 = K_0(\phi, b) > 1$  such that if  $1 < \lambda b < K_0$ , then the graph of  $f_{\lambda, b}^\phi$  has Hausdorff dimension  $D$ .*

Recently, Barański, Bárány and Romanowska [2], based on results of Ledrappier [10] and Tsujii [16], proved that for each integer  $b \geq 2$ , there is a number  $\lambda_b \in (0, 1)$  such that the Hausdorff dimension of the graph of  $W_{\lambda, b}$  is equal to  $D$  provided that  $\lambda_b < \lambda < 1$ . Furthermore, given an integer  $b \geq 2$ , they proved that the graph of  $f_{\lambda, b}^\phi$  has Hausdorff dimension  $D$  for generic  $(\lambda, \phi)$ . We refer to [2] for other progress on this and related problems. In order to prove our theorems, we have to introduce and verify a modified version of a transversality condition in [16] for all the cases. The proof of Theorem 1.1 also uses some results of [1].

The assumption that  $b$  is an integer enables us to approach the problem using dynamical systems theory. Indeed, in this case, the graph of  $f_{\lambda, b}^\phi$  can be interpreted as an invariant repeller for the expanding dynamical system  $\Phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ ,

$$\Phi(x, y) = \left( bx \mod 1, \frac{y - \widehat{\phi}(x)}{\lambda} \right),$$

where  $\widehat{\phi} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  denote the map naturally induced by  $\phi$ . By method of ergodic theory of smooth dynamical systems, Ledrappier [10] reduced the problem on Hausdorff dimension of the graph of  $f_{\lambda, b}^\phi$  to the study of local dimension of the measures  $m_x$  defined below.

Let  $\mathcal{A} = \{0, 1, \dots, b-1\}$ , and consider the Bernoulli measure  $\mathbb{P}$  on  $\mathcal{A}^{\mathbb{Z}^+}$  with uniform probabilities  $\{1/b, 1/b, \dots, 1/b\}^{\mathbb{Z}^+}$ . For  $x \in \mathbb{R}$  and  $\mathbf{u} = \{u_n\}_{n=1}^\infty \in \mathcal{A}^{\mathbb{Z}^+}$ , define

$$(1.1) \quad S(x, \mathbf{u}) = \sum_{n=1}^{\infty} \gamma^{n-1} \psi \left( \frac{x}{b^n} + \frac{u_1}{b^n} + \dots + \frac{u_n}{b^n} \right),$$

where

$$(1.2) \quad \gamma = \frac{1}{\lambda b} \text{ and } \psi(x) = \phi'(x).$$

These functions are, up to some multiplicative constant, the slope of the strong unstable manifolds of the expanding endomorphism  $\Phi$ . For each  $x \in \mathbb{R}$ , let  $m_x$  denote the Borel probability measure in  $\mathbb{R}$  obtained as pushforward of the measure  $\mathbb{P}$  by the function  $\mathbf{u} \mapsto S(x, \mathbf{u})$ .

We say that a Borel measure  $\mu$  in a metric space  $X$  has local dimension  $d$  at a point  $x \in X$ , if

$$\lim_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r} = d,$$

where  $B_r(x)$  denotes the ball of radius  $r$  centered at  $x$ . If the local dimension of  $\mu$  exists and is equal to  $d$  at  $\mu$ -a.e.  $x$ , then we say that  $\mu$  has local dimension  $d$  and write  $\dim(\mu) = d$ . It is well-known that if  $\mu$  has local dimension  $d$ , then any Borel set of positive measure has Hausdorff dimension at least  $d$ .

**Ledrappier's Theorem.** *Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous, piecewise  $C^{1+\alpha}$  and  $\mathbb{Z}$ -periodic function. Assume that  $\dim(m_x) = 1$  holds for Lebesgue a.e.  $x \in (0, 1)$ . Then the Hausdorff dimension of the graph of  $f_{\lambda, b}^\phi$  is equal to  $D$ .*

To prove this theorem, Ledrappier studied the local dimension of the measure  $\mu = \mu_{\lambda, b}^\phi$  obtained as the lift of the Lebesgue measure on  $[0, 1]$  to the graph of  $f_{\lambda, b}^\phi$ . Combining results of Ledreppier and Young [11] with a variation of Marstrand's projection theorem, Ledrappier proved that  $\dim(\mu) = D$ , provided that  $\dim(m_x) = 1$  holds for Lebesgue almost every  $x$ . This proves that the Hausdorff dimension of the graph of  $f_{\lambda, b}^\phi$  is at least  $D$ . As it is easy to see that the box dimension is at most  $D$ , the theorem follows. For the convenience of the readers not familiar with [11], we include a self-contained elementary proof of Ledrappier's Theorem in the appendix (assuming  $\phi'$  has no discontinuity for simplicity). The proof is of course motivated by the original proof in [10], but we also borrowed ideas in [9] where Keller gives an alternative proof of a weak version of Ledrappier's theorem. Keller's version is indeed enough for our purpose, although he used notation quite different from us.

Clearly, if  $m_x$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ , then  $\dim(m_x) = 1$ . The case when  $\phi(x) = d(x, \mathbb{Z})$  and  $b = 2$  is a famous problem in harmonic analysis and was studied first in [4]. In this case, the absolute continuity of  $m_x$  was established in [15] for almost every  $\gamma \in (1/2, 1)$ . See also [13]. In general,  $m_x$ 's are the conditional measures along vertical fibers of the unique SRB measure  $\vartheta = \vartheta_{b, \gamma}^\psi$  of the skew product map  $T : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ ,

$$(1.3) \quad T(x, y) = \left( bx \mod 1, \gamma y + \widehat{\psi}(x) \right),$$

where  $\psi(x)$  and  $\gamma$  are as above. The map  $T$  is an Anosov endomorphism and uniformly contracting along vertical fibers. The graph of the functions  $x \mapsto S(x, \mathbf{u})$  are the unstable manifolds. In [16], Tsujii posed some condition on the transversality of these unstable manifolds and showed that this condition implies absolute continuity of  $m_x$  for almost every  $x$  (and the absolute continuity of the SRB measure  $\vartheta$ ). Furthermore, for given  $b$ , he verified his condition for generic  $(\gamma, \psi)$ .

However, for given  $\psi$  it is not easy to verify Tsujii's condition, if possible at all. In fact, it was a major step in the recent work [2] to verify that Tsujii's condition holds for  $\psi(x) = -2\pi \sin(2\pi x)$  when  $\lambda \in (\lambda_b, 1)$ . We shall show in Section 3 that Tsujii's condition is indeed satisfied when  $b \geq 6$  for this particular  $\psi$  and all  $\lambda \in (1/b, 1)$  (or equivalently, all  $\gamma \in (1/b, 1)$ ). To deal with the case  $2 \leq b \leq 5$ , we shall pose a modified version of Tsujii's condition. We shall show that the new (weaker) condition is still enough to guarantee absolute continuity of  $m_x$  for Lebesgue a.e.  $x$ . Then we verify this new condition and conclude the proof of the Main Theorem by Ledrappier's Theorem.

**Theorem 1.2.** *Let  $b \geq 2$  be an integer, let  $\gamma \in (1/b, 1)$  and let  $\psi = -2\pi \sin(2\pi x)$ . Then the SRB measure  $\vartheta$  for the map  $T$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$  and with a square integrable density. In particular, for Lebesgue a.e.  $x \in \mathbb{R}$ , the measure  $m_x$  defined above is absolutely continuous with respect to Lebesgue measure and with a square integrable density.*

In the next section, we modify Tsujii's transversality condition. In particular, we shall define a new number  $\sigma(q)$  to replace the number  $e(q)$  in Tsujii's work. We shall prove Theorem 1.1 and state the plan of the proof of Theorem 1.2 in that section. Sections 2-5 are devoted to the proof of Theorem 1.2. In the appendix, Section 6, we provide a proof of Ledrappier's theorem.

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## 2. TSUJII'S TRANSVERSALITY CONDITION ON FAT SOLENOIDAL ATTRACTORS

In this section, we study a map  $T$  of the form (1.3), where  $b \geq 2$  is an integer,  $b^{-1} < \gamma < 1$  and  $\psi$  is a  $\mathbb{Z}$ -periodic  $C^1$  function. These maps were studied in [16] from measure-theoretical point of view, and in [1] from topological point of view. In [16], Section 2, it was shown that  $T$  has a unique SRB measure  $\vartheta$ , for which Lebesgue almost every point  $(x, y)$  in  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$  is a generic point, i.e.

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x,y)} \rightarrow \vartheta \text{ as } n \rightarrow \infty,$$

in the weak star topology, where  $\delta$  denote the Dirac measure. The measure  $\vartheta$  has an explicit expression through the measures  $m_x$  defined in the introduction: identifying  $\mathbb{R}/\mathbb{Z}$  with  $[0, 1)$  in the natural way, for each Borel set  $B \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ ,

$$\vartheta(B) = \int_0^1 m_x(B_x) dx,$$

where  $B_x = \{y \in \mathbb{R} : (x, y) \in B\}$ . We are interested in the absolute continuity of the SRB measure  $\vartheta$ , or equivalently, the absolute continuity of  $m_x$  for Lebesgue almost every  $x$ . In [16], Tsujii posed some condition on the transversality of the graphs of the functions  $S(x, \mathbf{u})$  (which are understood as unstable manifolds of  $T$ ) which guarantees the absolute continuity of  $\vartheta$ .

In this section, we introduce a modified version of Tsujii's condition and show that the weaker condition already implies absolute continuity of  $\vartheta$ . We shall prove Theorem 1.1 by verifying the modified condition.

**Notation.** For each  $x \in \mathbb{R}$  and  $(u_1 u_2 \cdots u_q) \in \mathcal{A}^q$ , let

$$x(\mathbf{u}) = \frac{x + u_1 + u_2 b + \cdots + u_q b^{q-1}}{b^q}.$$

We use  $S'(x, \mathbf{u})$  to denote the derivative of  $S(x, \mathbf{u})$  regarded as a function of  $x$ .

**2.1. Transversality.** We say that two words  $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{Z}^+}$  are  $(\varepsilon, \delta)$ -transversal at a point  $x_0 \in \mathbb{R}$  if one of the following holds:

$$|S(x_0, \mathbf{i}) - S(x_0, \mathbf{j})| > \varepsilon \text{ or } |S'(x_0, \mathbf{i}) - S'(x_0, \mathbf{j})| > \delta.$$

Otherwise, we say that  $\mathbf{i}$  and  $\mathbf{j}$  are  $(\varepsilon, \delta)$ -tangent at  $x_0$ . Let  $E(q, x_0; \varepsilon, \delta)$  denote the set of pairs  $(\mathbf{k}, \mathbf{l}) \in \mathcal{A}^q \times \mathcal{A}^q$  for which there exist  $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$  such that  $\mathbf{k}\mathbf{u}$  and  $\mathbf{l}\mathbf{v}$  are  $(\varepsilon, \delta)$ -tangent at  $x_0$ . Let

$$E(q, x_0) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, x_0; \varepsilon, \delta)$$

and

$$e(q, x_0) = \max_{\mathbf{k} \in \mathcal{A}^q} \# \{ \mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0) \}.$$

For  $J \subset \mathbb{R}$ , define

$$E(q, J; \varepsilon, \delta) = \bigcup_{x_0 \in J} E(q, x_0; \varepsilon, \delta),$$

$$E(q, J) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, J; \varepsilon, \delta)$$

and

$$e(q, J) = \max_{\mathbf{k} \in \mathcal{A}^q} \# \{ \mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, J) \}.$$

Tsujii's notation  $e(q)$  is defined as

$$e(q) = \lim_{p \rightarrow \infty} \max_{k=0}^{b^p-1} e \left( q, \left[ \frac{k}{b^p}, \frac{k+1}{b^p} \right] \right).$$

The following was proved in [16], see Proposition 8 in Section 4.

**Theorem 2.1** (Tsujii). *If there exists a positive integer  $q$  such that  $e(q) < (\gamma b)^q$ , then the SRB measure  $\vartheta$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$  with square integrable density. In particular, for Lebesgue a.e.  $x \in [0, 1)$ ,  $m_x$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$  and with square integrable density.*

*Remark.* It is obvious that  $e(q) \geq e(q, x_0)$  for all  $x_0 \in [0, 1)$ . Indeed, by Proposition 2.2 and Lemma 2.3, one can prove  $e(q) = \max_{x \in [0, 1)} e(q, x) = \max_{x \in \mathbb{R}} e(q, x)$ , although we do not need this fact.

We are going to define  $\sigma(q)$ . Let us say that a measurable function  $\omega : [0, 1) \rightarrow (0, \infty)$  is a *weight function* if  $\|\omega\|_\infty < \infty$  and  $\|1/\omega\|_\infty < \infty$ . A *testing function of order  $q$*  is a measurable function  $V : [0, 1) \times \mathcal{A}^q \times \mathcal{A}^q \rightarrow [0, \infty)$ . A testing function of order  $q$  is called *admissible* if there exist  $\varepsilon > 0$  and  $\delta > 0$  such that the following hold: For any  $x \in [0, 1)$ , if  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$ , then

$$V(x, \mathbf{u}, \mathbf{v}) V(x, \mathbf{v}, \mathbf{u}) \geq 1.$$

So in particular, we have  $V(x, \mathbf{u}, \mathbf{u}) \geq 1$  for each  $x \in [0, 1)$  and each  $\mathbf{u} \in \mathcal{A}^q$ .

Given a weight function  $\omega$  and an admissible testing function  $V$  of order  $q$ , define a new measurable function  $\Sigma_{V, \omega}^q : [0, 1) \rightarrow \mathbb{R}$  as follows: For each  $x \in [0, 1)$ , let

$$\Sigma_{V, \omega}^q(x) = \sup \left\{ \frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathcal{A}^q \right\}.$$

Define

$$\sigma(q) = \inf \|\Sigma_{V, \omega}^q\|_\infty,$$

where the infimum is taken over all weight functions  $\omega$  and admissible testing functions  $V$  of order  $q$ . In § 2.2, we shall prove the following theorem:

**Theorem 2.2.** *If there exists an integer  $q \geq 1$  such that  $\sigma(q) < (\gamma b)^q$  then the SRB measure  $\vartheta$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$  with square integrable density. In particular, for Lebesgue a.e.  $x \in [0, 1)$ ,  $m_x$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$  and with square integrable density.*

The parameter  $\sigma(q)$  takes into account the fact that the number

$$\#\{\mathbf{v} : (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)\}$$

may depend on  $x$  and  $\mathbf{u}$  in a significant way. On the other hand, the parameter  $e(q)$  is the supremum of such numbers over all possible choices of  $x$  and  $\mathbf{u}$ .

**Lemma 2.1.**  $\sigma(q) \leq e(q)$ .

*Proof.* Fix  $\varepsilon, \delta > 0$ . Let  $\omega = 1$  be the constant weight function. For each  $x \in [0, 1)$ , define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 1, & \text{if } (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta); \\ 0, & \text{otherwise.} \end{cases}$$

Then for any  $x \in [0, 1)$  and  $\mathbf{u} \in \mathcal{A}^q$ , we have

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) = \#\{\mathbf{v} : (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)\}.$$

Thus

$$\sigma(q) \leq \|\Sigma_{V, \omega}\|_{\infty} \leq \sup_{x \in [0, 1), \mathbf{u} \in \mathcal{A}^q} \#\{\mathbf{v} : \mathbf{u}, \mathbf{v} \in E(q, x; \varepsilon, \delta)\}.$$

Letting  $\varepsilon, \delta \rightarrow 0$ , we obtain  $\sigma(q) \leq e(q)$ .  $\square$

The following proposition collects a few facts about the quantifiers in the transversality conditions.

**Proposition 2.2.** *For  $\mathbf{k}, \mathbf{l} \in \mathcal{A}^q$ , the following hold:*

- (1) *For any  $x_0 \in \mathbb{R}$ ,  $(\mathbf{k}, \mathbf{l}) \in E(q, x_0)$  if and only if there exist  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathcal{A}^{\mathbb{Z}^+}$  such that  $S(x, \mathbf{k}\mathbf{u}) - S(x, \mathbf{l}\mathbf{v})$  has a multiple zero at  $x_0$ .*
- (2) *If  $(\mathbf{k}, \mathbf{l}) \notin E(q, x_0)$ , then there is a neighborhood  $U$  of  $x_0$  and  $\varepsilon, \delta > 0$ , such that  $(\mathbf{k}, \mathbf{l}) \notin E(q, U; \varepsilon, \delta)$ .*
- (3) *For any compact  $K \subset \mathbb{R}$ , if  $(\mathbf{k}, \mathbf{l}) \notin E(q, K)$ , then there exist  $\varepsilon, \delta > 0$  such that  $(\mathbf{k}, \mathbf{l}) \notin E(q, K; \varepsilon, \delta)$ .*
- (4) *For any  $\varepsilon > \varepsilon' > 0, \delta > \delta' > 0$  there exists  $\eta > 0$  such that if  $|x - x_0| < \eta$ ,  $(\mathbf{k}, \mathbf{l}) \notin E(q, x_0; \varepsilon, \delta)$  then  $(\mathbf{k}, \mathbf{l}) \notin E(q, x; \varepsilon', \delta')$ .*

*Proof.* Let us endow  $\mathcal{A}^{\mathbb{Z}^+}$  with the usual product topology of the discrete topology on  $\mathcal{A}$ . Then  $\mathcal{A}^{\mathbb{Z}^+}$  is compact. Moreover, if  $\mathbf{u}^n \rightarrow \mathbf{u}$  in  $\mathcal{A}^{\mathbb{Z}^+}$ , then

$$S(x, \mathbf{u}^n) \rightarrow S(x, \mathbf{u}) \text{ and } S'(x, \mathbf{u}^n) \rightarrow S'(x, \mathbf{u})$$

uniformly as  $n \rightarrow \infty$ .

(1) The “if” part is obvious. For the “only if” part, assume  $(\mathbf{k}, \mathbf{l}) \in E(q, x_0)$ . Then for any  $n = 1, 2, \dots$ ,  $(\mathbf{k}, \mathbf{l}) \in E(q, x_0; 1/n, 1/n)$ , and so there exist  $\mathbf{u}^n, \mathbf{v}^n \in \mathcal{A}^{\mathbb{Z}^+}$  such that

$$|S(x_0, \mathbf{k}\mathbf{u}^n) - S(x_0, \mathbf{l}\mathbf{v}^n)| \leq 1/n, \text{ and } |S'(x_0, \mathbf{k}\mathbf{u}^n) - S'(x_0, \mathbf{l}\mathbf{v}^n)| \leq 1/n.$$

After passing to a subsequence, we may assume  $\mathbf{u}^n \rightarrow \mathbf{u}$  and  $\mathbf{v}^n \rightarrow \mathbf{v}$  in  $\mathcal{A}^{\mathbb{Z}^+}$  as  $n \rightarrow \infty$ . Then

$$S(x_0, \mathbf{k}\mathbf{u}) - S(x_0, \mathbf{l}\mathbf{v}) = S'(x_0, \mathbf{k}\mathbf{u}) - S'(x_0, \mathbf{l}\mathbf{v}) = 0.$$

(2) Arguing by contradiction, assume that the statement is false. Then there exists  $\{x_n\}_{n=1}^\infty$  such that  $x_n \rightarrow x_0$  and  $(\mathbf{k}, \mathbf{l}) \in E(q, x_n; 1/n, 1/n)$ . Thus there exist  $\mathbf{u}^n, \mathbf{v}^n \in \mathcal{A}^{\mathbb{Z}^+}$  such that

$$|S(x_n, \mathbf{k}\mathbf{u}^n) - S(x_n, \mathbf{l}\mathbf{v}^n)| \leq 1/n, \text{ and } |S'(x_n, \mathbf{k}\mathbf{u}^n) - S'(x_n, \mathbf{l}\mathbf{v}^n)| \leq 1/n.$$

After passing to a subsequence we may assume  $\mathbf{u}^n \rightarrow \mathbf{u}$ ,  $\mathbf{v}^n \rightarrow \mathbf{v}$ . It follows that

$$S(x_0, \mathbf{k}\mathbf{u}) - S(x_0, \mathbf{l}\mathbf{v}) = S'(x_0, \mathbf{k}\mathbf{u}) - S'(x_0, \mathbf{l}\mathbf{v}) = 0,$$

a contradiction.

(3) follows from (2).

(4) Since  $\psi$  is  $\mathbb{Z}$ -periodic and  $C^1$ , for any  $\xi > 0$  there exists  $\eta > 0$  such that if  $|x_1 - x_2| < \eta$ , then  $|\psi(x_1) - \psi(x_2)| < \xi$  and  $|\psi'(x_1) - \psi'(x_2)| < \xi$ . Then for any  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$ , we have

$$|S(x_1, \mathbf{u}) - S(x_2, \mathbf{u})| \leq \xi/(1 - \gamma),$$

$$|S'(x_1, \mathbf{u}) - S'(x_2, \mathbf{u})| \leq \xi/(b - \gamma).$$

The statement follows.  $\square$

We shall also use the following symmetry of the functions  $S(x, \mathbf{u})$ .

**Lemma 2.3.** *For any  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$ ,  $x \in \mathbb{R}$  and  $q \in \mathbb{Z}^+$ , we have  $e(q, x + 1) = e(q, x)$  and  $m_{x+1} = m_x$ .*

*Proof.* Indeed, for any  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$  and  $x \in \mathbb{R}$ , we have

$$S(x + 1, \mathbf{u}) = S(x, \text{add}(\mathbf{u})),$$

where  $\text{add} : \mathcal{A}^{\mathbb{Z}^+} \rightarrow \mathcal{A}^{\mathbb{Z}^+}$  the adding machine which can be defined as follows: Given  $\mathbf{u} = \{u_n\}_{n=1}^\infty \in \mathcal{A}^{\mathbb{Z}^+}$ , defining inductively  $v_n, w_n \in \mathcal{A}$  with the following properties:

- $w_1 = 1$ ;
- If  $u_n + w_n < b$  then  $v_n = u_n + w_n$  and  $w_{n+1} = 0$ ; otherwise, define  $v_n = 0$  and  $w_n = 1$ ,

then  $\text{add}(\mathbf{u}) = \{v_n\}_{n=1}^\infty$ . This is a homeomorphism of  $\mathcal{A}^{\mathbb{Z}^+}$  which preserves the Bernoulli measure  $\mathbb{P}$ . Thus  $m_{x+1} = m_x$ .

Since the first  $q$  elements of  $\text{add}(\mathbf{u})$  depend only on the first  $q$  element of  $\mathbf{u}$ ,  $\text{add}$  induces a bijection from  $\mathcal{A}^q$  onto itself, denoted also by  $\text{add}$ . By definition,  $(\mathbf{k}, \mathbf{l}) \in E(q, x + 1)$  if and only if  $(\text{add}(\mathbf{k}), \text{add}(\mathbf{l})) \in E(q, x)$ . Thus  $e(q, x + 1) = e(q, x)$ .  $\square$

**2.2. Proof of Theorem 2.2.** The proof of Theorem 2.2 is an easy modification of Tsujii's proof of Theorem 2.1. Fix a weight function  $\omega$  and an admissible testing function  $V$  of order  $q$  such that

$$\|\Sigma_{V, \omega}\|_\infty < (\gamma b)^q.$$

By definition, there exist  $\varepsilon, \delta > 0$  such that for any  $x \in [0, 1)$ , if  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$ , then

$$V(x, \mathbf{u}, \mathbf{v})V(x, \mathbf{v}, \mathbf{u}) \geq 1.$$

For Borel measures  $\rho$  and  $\rho'$  on  $\mathbb{R}$  any  $r > 0$ , let

$$(\rho, \rho')_r = \int_{\mathbb{R}} \rho(B(y, r))\rho'(B(y, r))dy,$$

and let

$$\|\rho\|_r = \sqrt{(\rho, \rho)_r}.$$

For a Borel subset  $J \subset \mathbb{R}$ , define

$$I_r(J) = \frac{1}{r^2} \int_J \omega(x) \|m_x\|_r^2 dx$$

and write  $I_r = I_r([0, 1])$ .

According to Lemma 4 of [16],  $\liminf_{r \rightarrow 0} \|\rho\|_r < \infty$  implies that  $\rho$  is absolutely continuous with respect to the Lebesgue measure and the density function is square integrable. Consequently, if  $\liminf I_r < \infty$ , then the conclusion of the theorem holds.

With slight abuse of language, let  $T^q(m_x)$  denote the pushforward of the measure  $m_x$  under the map  $y \mapsto \pi_2 \circ T^q(x, y)$ , where  $\pi_2(x, y) = y$ . Then

$$m_x = \frac{1}{b^q} \sum_{\mathbf{i} \in \mathcal{A}^q} T^q(m_{x(\mathbf{i})}).$$

Thus

$$\|m_x\|_r^2 = b^{-2q} \sum_{\mathbf{i}, \mathbf{j}} (T^q m_{x(\mathbf{i})}, T^q m_{x(\mathbf{j})})_r.$$

Let

$$I_r^o(J) = \frac{1}{b^{2q} r^2} \int_J \omega(x) \sum_{(\mathbf{i}, \mathbf{j}) \notin E(q, x; \varepsilon, \delta)} (T^q(m_{x(\mathbf{i})}), T^q(m_{x(\mathbf{j})}))_r dx$$

and

$$I_r^*(J) = \frac{1}{b^{2q} r^2} \int_J \omega(x) \sum_{(\mathbf{i}, \mathbf{j}) \in E(q, x; \varepsilon, \delta)} (T^q(m_{x(\mathbf{i})}), T^q(m_{x(\mathbf{j})}))_r dx.$$

Then

$$I_r(J) = I_r^o(J) + I_r^*(J).$$

We shall also write  $I_r^o = I_r^o([0, 1])$  and  $I_r^* = I_r^*([0, 1])$ .

**Lemma 2.4.** *There exists  $C > 0$  such that  $I_r^o \leq C$  holds for all  $r > 0$ .*

*Proof.* Fix  $\varepsilon' \in (0, \varepsilon)$  and  $\delta' \in (0, \delta)$ . By Proposition 2.2 (4), there exists a positive integer  $p$ , such that if  $x \in J_{p,k} := [k/b^p, (k+1)/b^p]$  and  $(\mathbf{i}, \mathbf{j}) \notin E(q, x; \varepsilon, \delta)$ , then  $(\mathbf{i}, \mathbf{j}) \notin E(q, J_{p,k}; \varepsilon', \delta')$ . It follows that

$$I_r^o \leq \frac{\|\omega\|_\infty}{b^{2q} r^2} \sum_{k=0}^{b^p-1} \int_{J_{p,k}} \sum_{(\mathbf{i}, \mathbf{j}) \notin E(q, J_{p,k}; \varepsilon', \delta')} (T^q(m_{x(\mathbf{i})}), T^q(m_{x(\mathbf{j})}))_r dx.$$

In Proposition 6 in [16], it was proved that there exists  $C' = C'(p, \varepsilon', \delta') > 0$  such that if  $(\mathbf{i}, \mathbf{j}) \notin E(q, J_{p,k}; \varepsilon', \delta')$ , then

$$\int_{J_{p,k}} (T^q(m_{x(\mathbf{i})}), T^q(m_{x(\mathbf{j})}))_r dx \leq C' r^2.$$

Thus  $I_r^o \leq C$ . □

In order to estimate the terms  $I_r^*(J)$ , Tsujii observed

**Lemma 2.5.** *For any  $\mathbf{i} \in \mathcal{A}^q$  and any  $x \in \mathbb{R}$ , we have*

$$\|T^q(m_{x(\mathbf{i})})\|_r^2 = \gamma^q \|m_{x(\mathbf{i})}\|_{\gamma^{-q} r}^2.$$



*Proof.* This follows immediately from the fact that  $T^q$  is a contraction of rate  $\gamma^q$  in the vertical direction.  $\square$

**Lemma 2.6.** *For each  $r > 0$ , we have*

$$I_r^* \leq \frac{\|\Sigma_{V,\omega}\|_\infty}{(b\gamma)^q} I_{\gamma^{-q}r}.$$

*Proof.* Let us first prove that for each  $x \in [0, 1)$ ,

(2.1)

$$\sum_{(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)} (T^q m_x(\mathbf{u}), T^q m_x(\mathbf{v}))_r \leq \gamma^q \sum_{\mathbf{u} \in \mathcal{A}^q} \left( \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) \right) \|m_x(\mathbf{u})\|_{r/\gamma^q}^2.$$

To this end, let  $\mathbf{u}_k, k = 1, 2, \dots, b^q$  be all the elements of  $\mathcal{A}^q$ . Fix  $x \in [0, 1)$  and prepare the following notation:  $V_{kl} = V(x, \mathbf{u}_k, \mathbf{u}_l)$ ,  $x_k = x(\mathbf{u}_k)$  and

$$\theta_{kl} = \begin{cases} 1 & \text{if } (\mathbf{u}_k, \mathbf{u}_l) \in E(q, x; \varepsilon, \delta) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)} (T^q m_x(\mathbf{u}), T^q m_x(\mathbf{v}))_r = \sum_{k=1}^{b^q} \|T^q m_{x_k}\|_r^2 + 2 \sum_{1 \leq k < l \leq b^q} \theta_{kl} (T^q m_{x_k}, T^q m_{x_l})_r.$$

For each  $1 \leq k < l \leq b^q$ , by the Cauchy-Schwarz inequality,

$$(T^q m_{x_k}, T^q m_{x_l})_r \leq \|T^q m_{x_k}\|_r \|T^q m_{x_l}\|_r.$$

Thus

$$2\theta_{kl} (T^q m_{x_k}, T^q m_{x_l})_r \leq V_{kl} \|T^q m_{x_k}\|_r^2 + V_{lk} \|T^q m_{x_l}\|_r^2.$$

Indeed, this is trivial if  $\theta_{kl} = 0$ , while if  $\theta_{kl} = 1$ , it follows from the previous inequality and  $V_{kl}V_{lk} \geq 1$ . Consequently,

$$\begin{aligned} 2 \sum_{1 \leq k < l \leq b^q} \theta_{kl} (T^q m_{x_k}, T^q m_{x_l})_r &\leq \sum_{1 \leq k < l \leq b^q} (V_{kl} \|T^q m_{x_k}\|_r^2 + V_{lk} \|T^q m_{x_l}\|_r^2) \\ &= \sum_{k=1}^{b^q} \left( \sum_{\substack{1 \leq l \leq b^q \\ l \neq k}} V_{kl} \right) \|T^q m_{x_k}\|_r^2, \end{aligned}$$

and hence

$$\sum_{(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)} (T^q m_x(\mathbf{u}), T^q m_x(\mathbf{v}))_r \leq \sum_{k=1}^{b^q} \left( \sum_{l=1}^{b^q} V_{kl} \right) \|T^q m_{x_k}\|_r^2.$$

By Lemma 2.5, the inequality (2.1) follows.

Multiplying  $\omega(x)$  on both sides of (2.1), we obtain

$$\omega(x) \sum_{(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)} (T^q m_x(\mathbf{u}), T^q m_x(\mathbf{v}))_r \leq \gamma^q \sum_{\mathbf{u} \in \mathcal{A}^q} \Sigma_{V,\omega}(x) \omega(x(\mathbf{u})) \|m_x(\mathbf{u})\|_{\gamma^{-q}r}^2.$$

Dividing both side by  $b^{2q}r^2$  and integrating over  $[0, 1)$ , we obtain

$$I_r^* \leq \frac{\|\Sigma_{V,\omega}\|_\infty}{b^{2q}\gamma^q} \frac{1}{(\gamma^{-q}r)^2} \sum_{\mathbf{u} \in \mathcal{A}^q} \int_0^1 \|m_x(\mathbf{u})\|_{\gamma^{-q}r}^2 \omega(x(\mathbf{u})) dx.$$

Let  $J(\mathbf{u}) = \{x(\mathbf{u}) : 0 \leq x < 1\}$ . Then

$$\int_0^1 \|m_x(\mathbf{u})\|_{\gamma^{-q}r}^2 \omega(x(\mathbf{u})) dx = b^q \int_{J(\mathbf{u})} \|m_x\|_{\gamma^{-q}r}^2 \omega(x) dx.$$

Since  $J(\mathbf{u})$ ,  $\mathbf{u} \in \mathcal{A}^q$ , form a partition of  $[0, 1)$ , it follows that

$$I_r^* \leq \frac{\|\Sigma_{V,\omega}\|_\infty}{(\gamma b)^q} \frac{1}{(\gamma^{-q}r)^2} \int_0^1 \|m_x\|_{\gamma^{-q}r}^2 \omega(x) dx = \frac{\|\Sigma_{V,\omega}\|_\infty}{(\gamma b)^q} I_{\gamma^{-q}r}.$$

□

*Completion of proof of Theorem 2.2.* By Lemma 2.4 and Lemma 2.6, there exists a constant  $C > 0$  such that

$$I_r = I_r^o + I_r^* \leq C + \beta I_{\gamma^{-q}r},$$

holds for all  $r > 0$ , where  $\beta = \|\Sigma_{V,\omega}\|_\infty / (\gamma b)^q \in (0, 1)$ . As  $I_r < \infty$  for each  $r > 0$ , it follows that  $\liminf_{r \searrow 0} I_r < \infty$ . By the remarks at the beginning of this subsection, the conclusion of the theorem follows. □

**2.3. Proof of Theorem 1.1.** In this subsection, we shall prove Theorem 1.1 using Theorem 2.2.

**Lemma 2.7.** *Suppose that for each  $x \in [0, 1)$ ,  $E(q, x) \neq \mathcal{A}^q \times \mathcal{A}^q$ . Then*

$$\sigma(q) \leq b^q - 2 + 2/\alpha,$$

where  $\alpha = \alpha(b, q) > 1$  satisfies

$$2 - \alpha = (b^q - 2)\alpha(\alpha - 1).$$

*Proof.* By Lemma 2.3, the assumption implies that for each  $x \in \mathbb{R}$ ,  $E(q, x) \neq \mathcal{A}^q \times \mathcal{A}^q$ . By Proposition 2.2 (2) and compactness of  $[0, 1]$ , there exists  $\varepsilon > 0, \delta > 0$  such that  $E(q, x; \varepsilon, \delta) \neq \mathcal{A}^q \times \mathcal{A}^q$  for each  $x \in [0, 1]$ . So we can find measurable functions  $\mathbf{k}, \mathbf{l} : [0, 1) \rightarrow \mathcal{A}^q$  such that  $(\mathbf{k}(x), \mathbf{l}(x)) \notin E(q, x; \varepsilon, \delta)$ . Define  $\omega(x) = 1$  for all  $x \in [0, 1)$ . Define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 1 & \text{if } \mathbf{u}, \mathbf{v} \notin \{\mathbf{k}(x), \mathbf{l}(x)\} \text{ or } \mathbf{u} = \mathbf{v}; \\ 0 & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{k}(x), \mathbf{l}(x)) \text{ or } (\mathbf{l}(x), \mathbf{k}(x)); \\ \alpha & \text{if } \mathbf{u} \in \{\mathbf{k}(x), \mathbf{l}(x)\} \text{ but } \mathbf{v} \notin \{\mathbf{k}(x), \mathbf{l}(x)\}; \\ \alpha^{-1} & \text{if } \mathbf{u} \notin \{\mathbf{k}(x), \mathbf{l}(x)\} \text{ but } \mathbf{v} \in \{\mathbf{k}(x), \mathbf{l}(x)\}. \end{cases}$$

Then  $V$  is an admissible test function of order  $q$ . For every  $x \in [0, 1)$ , if  $\mathbf{u} \notin \{\mathbf{k}(x), \mathbf{l}(x)\}$ , then

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) = b^q - 2 + \frac{2}{\alpha},$$

and if  $\mathbf{u} \in \{\mathbf{k}(x), \mathbf{l}(x)\}$ , then

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) = 1 + (b^q - 2)\alpha = b^q - 2 + \frac{2}{\alpha}.$$

Thus  $\sigma(q) \leq \|\Sigma_{V,\omega}\|_\infty \leq b^q - 2 + 2/\alpha$ . □

We shall use some results obtained in [1]. Fix an integer  $b \geq 2$ . We say that a  $\mathbb{Z}$ -periodic continuous function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is *cohomologous to 0* if there exists a continuous  $\mathbb{Z}$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\psi(x) = f(bx) - f(x)$  holds for all  $x \in \mathbb{R}$ . The main step is the following lemma.

**Lemma 2.8.** *Assume that  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a  $\mathbb{Z}$ -periodic  $C^1$  function that is not cohomologous to zero and  $\int_0^1 \psi(x)dx = 0$ . Then there exists  $\gamma_1 \in (0, 1)$  and a positive integer  $N$  such that if  $\gamma_1 < \gamma < 1$ , then  $E(N, x) \neq \mathcal{A}^N \times \mathcal{A}^N$  for each  $x \in \mathbb{R}$ .*

*Proof.* We shall prove that there exists  $\gamma_1$ ,  $N_1$  and  $x_1 \in \mathbb{R}$  such that  $E(N_1, x_1) \neq \mathcal{A}^{N_1} \times \mathcal{A}^{N_1}$ . Note that this is enough for the conclusion of this lemma. Indeed, let  $\mathbf{k}, \mathbf{l} \in \mathcal{A}^{N_1}$  be such that  $(\mathbf{k}, \mathbf{l}) \notin E(N_1, x_1)$ . Then by Proposition 2.2 (2), there exist  $\varepsilon > 0$  and  $\delta > 0$  a neighborhood  $U$  of  $x_1$  such that  $(\mathbf{k}, \mathbf{l}) \notin E(N_1, x_1; \varepsilon, \delta)$ . Let  $N_2$  be a positive integer such that  $b^{N_2}U + \mathbb{Z} = \mathbb{R}$ . Then for any  $y \in \mathbb{R}$ , there exists  $x \in U$  and  $k \in \mathbb{Z}$  such that  $y = b^{N_2}x + k$ . Since the words  $(00 \cdots 0\mathbf{k}), (00 \cdots 0\mathbf{l}) \in \mathcal{A}^{N_2+N_1}$  are transversal at  $b^{N_2}x$ , we have  $E(N_1 + N_2, y - k) \neq \mathcal{A}^{N_1+N_2} \times \mathcal{A}^{N_1+N_2}$  which is equivalent to  $E(N_1 + N_2, y) \neq \mathcal{A}^{N_1+N_2} \times \mathcal{A}^{N_1+N_2}$  by Lemma 2.3.

For each  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$ , let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$G(x, \mathbf{u}) = \sum_{n=1}^{\infty} \frac{1}{b^n} \psi' \left( \frac{x + u_1 + u_2 b + \cdots + u_n b^{n-1}}{b^n} \right).$$

Note that  $G(x) = G(x, \mathbf{0})$  satisfies the functional equation

$$bG(bx) = \psi'(x) + G(x).$$

We claim that  $G(x)$  is not  $\mathbb{Z}$ -periodic. Indeed, otherwise, from the equation above, we obtain  $\int_0^1 G(x)dx = 0$ . Then  $g(x) = \int_0^x G(t)dt$  defines a  $\mathbb{Z}$ -periodic function. Since  $\psi' = bg'(bx) - g'(x)$  and

$$\int_0^1 \psi(x)dx = \int_0^1 (g(bx) - g(x))dx = 0,$$

it follows that  $\psi(x) = g(bx) - g(x)$  holds for all  $x$ . This contradicts the assumption that  $\psi$  is not cohomologous to zero.

Since  $G(x+1) = G(x, (100 \cdots))$ , it follows that there exists  $x_1 \in [0, 1)$  such that

$$5\delta := |G(x_1, (000 \cdots)) - G(x_1, (100 \cdots))| > 0.$$

Let  $C = \max_{x \in [0, 1]} |\psi'(x)|$ . Let  $N_1$  be a positive integer such that

$$2C < \delta b^{N_1}(b-1)$$

and let  $\gamma_1 \in (0, 1)$  be such that

$$(1 - \gamma_1^{N_1})C < \delta(b-1).$$

Then, for any  $\mathbf{k}, \mathbf{l} \in \mathcal{A}^{\mathbb{Z}^+}$  with  $k_1 = k_2 = \cdots = k_{N_1} = 0$ ,  $l_1 = 1$ ,  $l_2 = l_3 = \cdots = l_{N_1} = 0$ , we have

$$\begin{aligned} \left| \frac{d}{dx} S(x_1, \mathbf{k}) - G(x_1) \right| &\leq \sum_{n=1}^{N_1} \frac{(1 - \gamma_1^{n-1})}{b^n} C + 2C \sum_{n=N_1+1}^{\infty} b^{-n} \\ &< (1 - \gamma_1^{N_1})C(b-1)^{-1} + 2C((b-1)b^{N_1})^{-1} < 2\delta, \end{aligned}$$

and similarly,

$$\left| \frac{d}{dx} S(x_1, \mathbf{l}) - G(x_1 + 1) \right| < 2\delta.$$

Therefore,

$$\left| \frac{d}{dx} S(x_1, \mathbf{k}) - \frac{d}{dx} S(x_1, \mathbf{l}) \right| \geq \delta.$$

It follows that the two words  $(00 \cdots 0), (10 \cdots 0) \in \mathcal{A}^{N_1}$  are transversal at  $x_1$ , hence

$$E(N_1, x_1) \neq \mathcal{A}^{N_1} \times \mathcal{A}^{N_1}.$$

□

*Proof of Theorem 1.1.* Let  $\psi = \phi'$ , so  $\psi$  is a  $\mathbb{Z}$ -periodic non-constant  $C^1$  function and  $\int_0^1 \psi(x) dx = 0$ . Consider the map  $T$  as in (1.3). By Ledrappier's Theorem, it suffices to prove the measures  $m_x$  defined for this map  $T$  are absolutely continuous for Lebesgue almost every  $x \in [0, 1]$ .

First we assume that  $\psi$  is not cohomologous to 0. By Lemma 2.8, there exists  $\gamma_1 \in (0, 1)$  and  $N$  such that if  $\gamma_1 < \gamma < 1$ , then  $E(N, x) \neq \mathcal{A}^N \times \mathcal{A}^N$  for each  $x \in \mathbb{R}$ . By Lemma 2.7, this implies that  $\sigma(N) < b^N - 2 + \frac{2}{\alpha}$  where  $\alpha = \alpha(b, N) \in (1, 2)$ . Thus there exists  $\gamma_0 \in (\gamma_1, 1)$  such that if  $\gamma > \gamma_0$  then  $\sigma(N) < (b\gamma)^N$ . By Theorem 2.2, it follows that  $m_x$  is absolutely continuous for Lebesgue a.e.  $x \in [0, 1]$ .

To complete the proof, we shall use a few results of [1]. Assume that  $\psi$  is cohomologous to 0 and let  $\psi_1 : \mathbb{R} \rightarrow \mathbb{R}$  be a  $\mathbb{Z}$ -periodic continuous function such that  $\psi_1(bx) - \psi_1(x) = \psi(x)$ . By Lemma 5.2 (5) and Lemma 5.8 (2) of that paper,  $\psi_1$  is  $C^1$ , and  $T_{b, \gamma, \psi}$  is  $C^1$  conjugate to  $T_{b, \gamma, \psi_1}$ . By adding a constant if necessary, we may assume  $\int_0^1 \psi_1(x) dx = 0$ . If  $\psi_1$  is not cohomologous to zero, then we are done. Otherwise, repeat the argument. By Lemma 5.6 of that paper, any  $\mathbb{Z}$ -periodic non-constant  $C^1$  function  $\psi$  is not infinitely cohomologous to zero. Thus the procedure stops within finitely many steps. □

**2.4. Plan of Proof of Theorem 1.2.** Theorem 1.2 follows from the following theorem by Theorem 2.2.

**Theorem 2.3.** *For an integer  $b \geq 2$ ,  $1/b < \gamma < 1$  and  $\psi(x) = -2\pi \sin(2\pi x)$ , consider the map  $T$  as in (1.3). Then there exists a positive integer  $q$  such that  $\sigma(q) < (b\gamma)^q$ .*

The rest of the paper is devoted to the proof of Theorem 2.3. The proof uses special property of the map  $\psi$  and breaks into several cases.

*Proof of Theorem 2.3.* The case  $b \geq 6$  is proved in Theorem 3.1 (i). The case  $b = 5$  is proved in Theorem 4.1. The case  $b = 4$  is proved in Theorem 4.2. The case  $b = 3$  is proved in Theorem 4.3. The case  $b = 2$  follows from Corollary 5.13 and Proposition 5.16. □

To conclude this section, we include a few lemmas which will be used in later sections. The first lemma is about a new symmetric property of the functions  $S(x, \mathbf{u})$  in the case that  $\psi$  is odd.

**Lemma 2.9** (Symmetry). *Assume that  $\psi(x)$  is an odd function. Then for any  $\mathbf{i} = \{i_n\}_{n=1}^\infty \in \mathcal{A}^{\mathbb{Z}^+}$ , letting  $\mathbf{i}' = \{i'_n\}_{n=1}^\infty$  with  $i'_n = b - 1 - i_n$ , we have*

$$-S(x, \mathbf{i}) = S(1 - x, \mathbf{i}').$$

*Proof.* This follows from the definition of  $S(\cdot, \cdot)$ . □

The next three lemmas will be used to obtain upper bounds for  $\sigma(q)$ .

**Lemma 2.10.** *Let  $q \geq 1$  be an integer. Suppose that there are constants  $\varepsilon > 0$  and  $\delta > 0$  and  $K \subset [0, 1)$  with the following properties:*

- (i) *For  $x \in K$ ,  $e(q, x; \varepsilon, \delta) = 1$  and for  $x \in [0, 1) \setminus K$ ,  $e(q, x; \varepsilon, \delta) \leq 2$ ;*
- (ii) *If  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$  for some  $x \in [0, 1) \setminus K$  and  $\mathbf{u} \neq \mathbf{v}$ , then both  $x(\mathbf{u})$  and  $x(\mathbf{v})$  belong to  $K$ .*

Then  $\sigma(q) \leq \sqrt{2}$ .

*Proof.* We define suitable weight function  $\omega$  and testing function  $V$ . Let  $L = [0, 1) \setminus K$ . Define

$$\omega(x) = \begin{cases} \sqrt{2} & \text{if } x \in K; \\ 1 & \text{if } x \in L. \end{cases}$$

Define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 1 & \text{if } (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta); \\ 0 & \text{otherwise.} \end{cases}$$

Then for  $x \in K$  and any  $\mathbf{u} \in \mathcal{A}^q$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) = \frac{\omega(x)}{\omega(x(\mathbf{u}))} \leq \sqrt{2}.$$

For  $x \in L$  and  $\mathbf{u} \in \mathcal{A}^q$ , if  $\mathbf{u}$  is not  $(\varepsilon, \delta)$ -tangent to any other element of  $\mathcal{A}^q$  at  $x$ , then we have

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) = \frac{\omega(x)}{\omega(x(\mathbf{u}))} \leq 1;$$

otherwise, we have  $\omega(x) = 1$  and  $\omega(x(\mathbf{u})) = \sqrt{2}$

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) = \frac{\omega(x)}{\omega(x(\mathbf{u}))} \cdot 2 \leq \sqrt{2}.$$

It follows that  $\sigma(q) \leq \Sigma_{V,w} \leq \sqrt{2}$ .  $\square$

**Lemma 2.11.** *Let  $q \geq 1$  be an integer. Suppose that there are constants  $\varepsilon > 0$  and  $\delta > 0$  and  $K \subset [0, 1)$  with the following properties:*

- (i) *For  $x \in K$ ,  $e(q, x; \varepsilon, \delta) \leq 1$  and for  $x \in [0, 1) \setminus K$ ,  $e(q, x; \varepsilon, \delta) \leq 2$ ;*
- (ii) *If  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$  for some  $x \in [0, 1) \setminus K$  and  $\mathbf{u} \neq \mathbf{v}$ , then either  $x(\mathbf{u}) \in K$  or  $x(\mathbf{v}) \in K$ .*

Then  $\sigma(q) \leq (\sqrt{5} + 1)/2$ .

*Proof.* Let  $L = [0, 1) \setminus K$ . Define

$$\omega(x) = \begin{cases} (\sqrt{5} + 1)/2 & \text{if } x \in K; \\ 1 & \text{otherwise.} \end{cases}$$

For  $x \in K$ , define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 1 & \text{if } \mathbf{u} = \mathbf{v} \\ 0 & \text{otherwise.} \end{cases}$$

For  $x \in L$ , define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 0 & \text{if } (\mathbf{u}, \mathbf{v}) \notin E(q, x; \varepsilon, \delta); \\ 1 & \text{if } \mathbf{u} = \mathbf{v}; \\ (\sqrt{5} + 1)/2 & \text{if } (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta), \mathbf{u} \neq \mathbf{v}, \text{ and } x(\mathbf{u}) \in K; \\ (\sqrt{5} - 1)/2 & \text{if } (\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta), \mathbf{u} \neq \mathbf{v}, \text{ and } x(\mathbf{u}) \notin K. \end{cases}$$

Then for  $x \in K$  and any  $\mathbf{u} \in \mathcal{A}^q$ , we have

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) \leq \frac{\sqrt{5}+1}{2}.$$

For  $x \in L$  and  $\mathbf{u} \in \mathcal{A}^q$ , if  $x(\mathbf{u}) \in K$ , we have

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) \leq \frac{1}{(\sqrt{5}+1)/2} \left( \frac{\sqrt{5}+1}{2} + 1 \right) = \frac{\sqrt{5}+1}{2};$$

if  $x(\mathbf{u}) \notin K$ , then

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v}} V(x, \mathbf{u}, \mathbf{v}) \leq \frac{1}{1} \left( \frac{\sqrt{5}-1}{2} + 1 \right) = \frac{\sqrt{5}+1}{2}.$$

In conclusion, we have  $\sigma(q) \leq \|\Sigma_{V,\omega}\|_{\infty} \leq (\sqrt{5}+1)/2$ .  $\square$

The next lemma is more technical and will only be used in the case  $b = 2$ .

**Lemma 2.12.** *Let  $q \geq 1$  be an integer. Suppose there are three pairwise disjoint subsets  $K_0, K_1, K_2$  of  $[0, 1)$  with  $K_0 \cup K_1 \cup K_2 = [0, 1)$  and constants  $\varepsilon, \delta > 0$  such that the following hold:*

- (i) *For each  $x \in K_0$ ,  $e(q, x; \varepsilon, \delta) = 1$ ;*
- (ii) *For each  $x \in K_1$ , there exist  $\mathbf{a}_x, \mathbf{b}_x \in \mathcal{A}^q$  such that  $x(\mathbf{a}_x), x(\mathbf{b}_x) \in K_0$  and such that  $(\mathbf{a}_x, \mathbf{b}_x)$  and  $(\mathbf{b}_x, \mathbf{a}_x)$  are the only possible non-trivial element of  $E(q, x; \varepsilon, \delta)$ ;*
- (iii) *For  $x \in K_2$ , there exist  $\mathbf{a}_x, \mathbf{b}_x, \mathbf{c}_x \in \mathcal{A}^q$  such that  $x(\mathbf{a}_x), x(\mathbf{b}_x) \in K_0$  and  $x(\mathbf{c}_x) \in K_1$  and such that  $(\mathbf{a}_x, \mathbf{b}_x)$ ,  $(\mathbf{a}_x, \mathbf{c}_x)$ ,  $(\mathbf{b}_x, \mathbf{a}_x)$  and  $(\mathbf{c}_x, \mathbf{a}_x)$  are the only possible non-trivial elements of  $E(q, x; \varepsilon, \delta)$ .*

Then

$$\sigma(q) \leq t < 1.61,$$

where  $t > \sqrt{2}$  is the unique solution of the following equation

$$(2.2) \quad \frac{1}{t^2-1} + \frac{2}{t^3-2} + 1 = t^2.$$

*Proof.* Let  $s = t^2/2$ . Note that  $t > s > 1$ . Define

$$\omega(x) = \begin{cases} t & \text{if } x \in K_0; \\ s & \text{if } x \in K_1; \\ 1 & \text{if } x \in K_2. \end{cases}$$

For  $x \in K_0 \cup K_1$ , define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 0 & \text{if } (\mathbf{u}, \mathbf{v}) \notin E(q, x; \varepsilon, \delta); \\ 1 & \text{otherwise.} \end{cases}$$

For  $x \in K_2$ , define

$$V(x, \mathbf{u}, \mathbf{v}) = \begin{cases} 1 & \text{if } \mathbf{u} = \mathbf{v}; \\ ts-1 & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{c}_x, \mathbf{a}_x); \\ (ts-1)^{-1} & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{a}_x, \mathbf{c}_x); \\ t^2-1 & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{b}_x, \mathbf{a}_x); \\ (t^2-1)^{-1} & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{a}_x, \mathbf{b}_x); \\ 0 & \text{otherwise.} \end{cases}$$

Then for  $x \in K_0$ , and any  $\mathbf{u} \in \mathcal{A}^q$  we have

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) \leq t;$$

for  $x \in K_1$ ,  $\mathbf{u} \in \{\mathbf{a}_x, \mathbf{b}_x\}$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) = 2s/t = t;$$

for  $x \in K_1$ ,  $\mathbf{u} \notin \{\mathbf{a}_x, \mathbf{b}_x\}$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) \leq s < t;$$

for  $x \in K_2$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{a}_x))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) = \frac{1}{t} \left( \frac{1}{t^2 - 1} + \frac{1}{ts - 1} + 1 \right) = t;$$

for  $x \in K_2$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{b}_x))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{b}_x, \mathbf{v}) = \frac{1}{t} (1 + t^2 - 1) = t;$$

for  $x \in K_2$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{c}_x))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{c}_x, \mathbf{v}) = \frac{1}{s} (1 + ts - 1) = t;$$

and for  $x \in K_2$ ,  $\mathbf{u} \notin \{\mathbf{a}_x, \mathbf{b}_x, \mathbf{c}_x\}$ ,

$$\frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) \leq 1.$$

Therefore

$$\sigma(q) \leq \|\Sigma_{V, \omega}(x)\|_{\infty} \leq t.$$

□

### 3. THE CASE WHEN $b$ IS LARGE

In this and next sections, we shall prove Theorem 2.3. So we consider a map  $T$  of the form (1.3) with  $\psi = -2\pi \sin(2\pi x)$ . The main result of this section is the following:

**Theorem 3.1.** (1) If  $b \geq 6$ , then  $\sigma(1) \leq e(1) < \gamma b$ .  
 (2) If  $b = 4, 5$ , then either  $e(1) = 2$  or  $e(1) < \gamma b$ .  
 (3) If  $b = 3$ , then either  $e(1) = 2$  or  $\sigma(1) < \gamma b$ .

We start with a few lemmas. Let

$$\Delta_{b, \gamma} = \max_{t \in \mathbb{R}} (\sin(bt) + \gamma \sin(t)).$$

Besides the trivial bound:  $\Delta_{b, \gamma} \leq 1 + \gamma$ , we also need the following:

**Lemma 3.1.** For each  $\gamma \in (0, 1)$ , we have

$$(3.1) \quad \Delta_{6, \gamma} \leq \max(1 + 0.972\gamma, 0.99 + \gamma)$$

$$(3.2) \quad \Delta_{3, \gamma} \leq 1 + 0.71\gamma.$$

*Proof.* Let us first prove (3.2). Indeed, if  $\sin t \leq 0.71$  then the inequality holds. So assume  $\sin t > 0.71$ . Then  $\sin(3t) = 3\sin t - 4\sin^3 t \leq 0.71$ , and hence  $\sin(3t) + \gamma \sin t \leq 0.71 + \gamma \leq 1 + 0.71\gamma$ .

Let us prove (3.1). If  $\sin t \leq 0.972$  then the inequality holds. So assume  $\sin t > 0.972$ , then  $\sin(3t) \leq 3 \cdot 0.972 - 4 \cdot 0.972^3 = -0.757$ . Then

$$|\sin(6t)| \leq 2|\sin(3t)|\sqrt{1 - \sin^2(3t)} < 0.99.$$

□

**Lemma 3.2.** *If  $(k, l) \in E(1, x^*)$ , then*

$$(3.3) \quad \left| \sin \frac{2\pi(x^* + k)}{b} - \sin \frac{2\pi(x^* + l)}{b} \right| \leq \frac{2\Delta_{b,\gamma}}{1 - \gamma^2} \leq \frac{2\gamma}{1 - \gamma},$$

$$(3.4) \quad \left| \cos \frac{2\pi(x^* + k)}{b} - \cos \frac{2\pi(x^* + l)}{b} \right| \leq \frac{2\gamma}{b - \gamma},$$

$$(3.5) \quad 4\sin^2 \frac{\pi(k - l)}{b} \leq \left( \frac{2\gamma\Delta_{b,\gamma}}{1 - \gamma^2} \right)^2 + \left( \frac{2\gamma}{b - \gamma} \right)^2 \leq \left( \frac{2\gamma}{1 - \gamma} \right)^2 + \left( \frac{2\gamma}{b - \gamma} \right)^2.$$

*Proof.* By Proposition 2.2 (1), there exists  $\mathbf{k} = \{k_n\}_{n=1}^\infty$  and  $\mathbf{l} = \{l_n\}_{n=1}^\infty$  in  $\mathcal{A}^{\mathbb{Z}^+}$  with  $k_1 = k$  and  $l_1 = l$  such that for  $F(x) := -(2\pi)^{-1}(S(x, \mathbf{k}) - S(x, \mathbf{l}))$ , we have  $F(x^*) = F'(x^*) = 0$ . Let  $f(x) = \sin(2\pi bx) + \gamma \sin(2\pi x)$ . Then

$$\left| -\frac{S(x, \mathbf{k})}{2\pi} - \sin \frac{2\pi(x + k)}{b} \right| \leq \sum_{n=1}^\infty \gamma^{2n-1} |f(x_{2n+1})| \leq \frac{2\gamma\Delta_{b,\gamma}}{1 - \gamma^2},$$

and

$$\left| -\frac{bS'(x, \mathbf{k})}{4\pi^2} - \cos \frac{2\pi(x + k)}{b} \right| \leq \sum_{n=2}^\infty \left( \frac{\gamma}{b} \right)^{n-1} |\cos(2\pi x_n)| \leq \frac{\gamma}{b - \gamma},$$

where  $x_n = (x + k_1 + k_2b + \cdots + k_nb^{n-1})/b^n$ . Similarly,

$$\left| -\frac{S(x, \mathbf{l})}{2\pi} - \sin \frac{2\pi(x + l)}{b} \right| \leq \frac{2\gamma\Delta_{b,\gamma}}{1 - \gamma^2},$$

and

$$\left| -\frac{bS'(x, \mathbf{l})}{4\pi^2} - \cos \frac{2\pi(x + l)}{b} \right| \leq \frac{\gamma}{b - \gamma}.$$

Therefore,

$$\begin{aligned} \left| F(x) - \left( \sin \frac{2\pi(x + k)}{b} - \sin \frac{2\pi(x + l)}{b} \right) \right| &\leq \frac{2\gamma\Delta_{b,\gamma}}{1 - \gamma^2}, \\ \left| \frac{bF'(x)}{2\pi} - \left( \cos \frac{2\pi(x + k)}{b} - \cos \frac{2\pi(x + l)}{b} \right) \right| &\leq \frac{2\gamma}{b - \gamma}. \end{aligned}$$

Substituting  $x = x^*$  gives us (3.3) and (3.4). The inequality (3.5) follows from these two inequalities and the following

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2 \frac{y - x}{2}.$$

□



Pick up  $z \in [0, 1]$  such that  $e(1, z) = e(1)$  and pick up  $k \in \{0, 1, \dots, b-1\}$  such that

$$\#\{l \in \{0, 1, \dots, b-1\} : (k, l) \in E(1, z)\} = e(1).$$

Let  $k_1, k_2, \dots, k_{e(1)}$  be all the elements in  $E(1, z)$ , arranged in such a way that

$$\sin(2\pi x_1) \leq \sin(2\pi x_2) \leq \dots \leq \sin(2\pi x_{e(1)}),$$

where  $x_i = (z + k_i)/b$ .

**Lemma 3.3.** *Under the above circumstances, the following holds:*

(1) *For each  $1 \leq i < j \leq e(1)$ , we have*

$$(3.6) \quad |\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \frac{2\theta_1(b, \gamma)}{b},$$

where

$$(3.7) \quad \theta_1(b, \gamma) = \sqrt{\max\left(0, \left(b \sin \frac{\pi}{b}\right)^2 - \frac{4\gamma^2 b^2}{(b-\gamma)^2}\right)}.$$

(2) *If  $k_i = k$  or  $k_j = k$ , then*

$$(3.8) \quad |\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \frac{2\theta_0(b, \gamma)}{b},$$

where

$$(3.9) \quad \theta_0(b, \gamma) = \sqrt{\max\left(0, \left(b \sin \frac{\pi}{b}\right)^2 - \frac{\gamma^2 b^2}{(b-\gamma)^2}\right)}.$$

(3) *If  $k_i - k_j \not\equiv \pm 1 \pmod{b}$ , then*

$$(3.10) \quad |\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \frac{2\theta_2(b, \gamma)}{b},$$

where

$$(3.11) \quad \theta_2(b, \gamma) = \sqrt{\max\left(0, \left(b \sin \frac{2\pi}{b}\right)^2 - \frac{4\gamma^2 b^2}{(b-\gamma)^2}\right)}.$$

*Proof.* For each  $1 \leq i < j \leq e(1)$ , we have

$$(3.12) \quad |\cos(2\pi x_i) - \cos(2\pi x_j)|^2 + |\sin(2\pi x_i) - \sin(2\pi x_j)|^2 = 4 \sin^2(\pi(x_i - x_j)) \geq 4 \sin^2 \frac{\pi}{b}.$$

If  $k_i = k$  or  $k_j = k$  then by (3.4), the inequality (3.8) follows.

In general, from (3.4), we obtain

$$(3.13) \quad |\cos(2\pi x_i) - \cos(2\pi x_j)| \leq \frac{4\gamma}{b-\gamma},$$

which, together with (3.12), implies (3.6).

If  $k_i - k_j \not\equiv \pm 1 \pmod{b}$ , then

$$(3.14) \quad |\cos(2\pi x_i) - \cos(2\pi x_j)|^2 + |\sin(2\pi x_i) - \sin(2\pi x_j)|^2 \geq 4 \sin^2 \frac{2\pi}{b},$$

which, together with (3.13), implies (3.10).  $\square$

**3.1. The case when  $b \geq 6$ .** We shall prove Theorem 3.1 in the case  $b \geq 6$ . We separate the argument in two propositions.

**Proposition 3.4.** *Assume  $b \geq 6$ . If there exists  $1 \leq i < e(1)$  such that  $k_{i+1} - k_i \not\equiv \pm 1 \pmod{b}$ , then  $e(1) < \gamma b$ .*

*Proof.* Under assumption, we have

$$\begin{aligned} \frac{2\theta_2(b, \gamma)}{b} + (e(1) - 2) \frac{2\theta_1(b, \gamma)}{b} &\leq \sum_{i=1}^{e(1)-1} (\sin(2\pi x_{i+1}) - \sin(2\pi x_i)) \\ &= \sin(2\pi x_{e(1)}) - \sin(2\pi x_1) \leq 2, \end{aligned}$$

and so

$$(3.15) \quad b \geq (e(1) - 2)\theta_1(\gamma, b) + \theta_2(\gamma, b).$$

We may assume

$$(3.16) \quad \gamma \leq \frac{b - \theta_2(b, \gamma) + 2\theta_1(b, \gamma)}{\theta_1(b, \gamma)b},$$

for otherwise we are done.

Note that  $t \mapsto \sin t/t$  is monotone decreasing in  $[0, \frac{\pi}{2})$ . Since  $b \geq 6$ , we have

$$\theta_1(6, \gamma) \geq \theta_1(6, 1) = 1.8 \text{ and } \theta_2(6, \gamma) \geq \theta_2(6, 1) = \sqrt{21.24} > 4.$$

By (3.16), it follows that

$$\gamma \leq \frac{b - 4 + 2 \cdot 1.8}{1.8b} < \frac{5}{9}.$$

Therefore, we have

$$(3.17) \quad \theta_1(6, \gamma) \geq \theta_1(6, 5/9) > 2.5 \text{ and } \theta_2(6, \gamma) \geq \theta_2(6, 5/9) > 5,$$

and hence

$$(3.18) \quad \gamma \leq \frac{b - 5 + 2 \cdot 2.5}{2.5b} = \frac{2}{5}.$$

Moreover, by (3.15),

$$(3.19) \quad b > 2.5e(1).$$

**Case 1.**  $e(1) \leq 3$ .

Indeed, this is clear if  $e(1) = 1$  as we assume  $\gamma b > 1$ . If  $e(1) = 2$  or  $3$ , then there exists  $i, j$  such that  $k_i = k \neq k_j$ , and

$$|\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \frac{2\theta_2(b, \gamma)}{b}.$$

On the other hand, since  $(k_i, k_j) \in E(1, z)$ , we have

$$|\sin(2\pi x_i) - \sin(2\pi x_j)| \leq \frac{2\gamma}{1 - \gamma}.$$

Therefore,

$$\frac{2\gamma}{1 - \gamma} \geq \frac{2\theta_2(b, \gamma)}{b},$$

which, together with (3.17) and (3.18), implies that

$$\gamma b \geq (1 - \gamma)\theta_2(b, \gamma) > 5(1 - 2/5) = 3 \geq e(1).$$

**Case 2.**  $e(1) \geq 4$ .

By (3.19), we have  $b > 2.5e(1) \geq 10$ . Since  $b$  is an integer, this implies  $b \geq 11$ . Thus  $\theta_1(b, \gamma) \geq \theta_1(11, 2/5) > 2.9$  and  $\theta_2(b, \gamma) \geq \theta_2(11, 2/5) > 5.8$ , where we use the numerics:  $\sin(\pi/11) > 0.28$  and  $\sin(2\pi/11) > 0.54$ . By (3.15) and (3.16), we obtain

$$(3.20) \quad b \geq 2.9e(1) \text{ and } \gamma \leq \frac{10}{29}.$$

Let us first consider the case  $e(1) = 4$ . Then by (3.20), we have  $b \geq 12$ , hence  $\theta_1(b, \gamma) \geq \theta_1(12, 10/29) > 3$ . On the other hand, there exists  $1 \leq i, j \leq 4$  such that  $k_i = k$  and  $|i - j| \geq 2$ . Thus

$$(3.21) \quad \frac{2\gamma}{1-\gamma} \geq |\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \frac{4\theta_1(\gamma, b)}{b}.$$

Therefore,  $\frac{2\gamma}{1-\gamma} > \frac{12}{b}$ , which implies that

$$\gamma b > 6(1 - \gamma).$$

If  $\gamma \leq 1/3$ , then it follows that  $\gamma b > 4 = e(1)$ . If  $\gamma > 1/3$ , then  $\gamma b > 12 \cdot (1/3) = 4 = e(1)$ .

Now let us assume  $e(1) \geq 5$ . Then by (3.20) we have  $b > 2.9e(1) \geq 14.5$  which implies that  $b \geq 15$ . Then

$$\theta_1(\gamma, b) \geq \theta_1(15, 10/29) > 3 \text{ and } \theta_2(\gamma, b) \geq \theta_2(15, 10/29) > 6.$$

By (3.16), it follows that  $\gamma < 1/3$ . Since

$$\frac{4\gamma}{1-\gamma} \geq |\sin(2\pi x_{e(1)}) - \sin(2\pi x_1)| \geq \frac{2\theta_1}{b}(e(1) - 2) + \frac{2\theta_2}{b} > \frac{6e(1)}{b}$$

we obtain

$$\gamma b > 6(1 - \gamma)e(1)/4 > e(1).$$

□

**Proposition 3.5.** *Assume  $b \geq 6$ . If  $k_{i+1} - k_i = \pm 1 \pmod{b}$  for each  $i \in \{1, 2, \dots, e(1) - 1\}$  then  $e(1) < \gamma b$ .*

*Proof.* For definiteness of notation, let us assume  $k_2 - k_1 = 1 \pmod{b}$ . Then since  $k_1, k_2, \dots, k_{e(1)} \in \{0, 1, \dots, b-1\}$ , we have  $k_{i+1} - k_i = 1 \pmod{b}$  for each  $1 \leq i < e(1)$ . Put  $y^i = \pi(2k_1 + 2i - 1 + 2z)/b$ . Then for each  $1 \leq i < e(1)$ ,

$$\cos(2\pi x_{i+1}) - \cos(2\pi x_i) = -2 \sin \frac{\pi}{b} \sin y^i,$$

and

$$0 \leq \sin(2\pi x_{i+1}) - \sin(2\pi x_i) = 2 \sin \frac{\pi}{b} \cos y^i.$$

By (3.13), it follows that

$$|\sin y^i| \leq \frac{2\gamma}{(b - \gamma) \sin(\pi/b)} < 0.8\gamma,$$

where we use

$$(b - \gamma) \sin(\pi/b) > (b - 1) \sin(\pi/b) \geq 5 \sin(\pi/6) = 5/2.$$

Therefore,

$$y^i \in \bigcup_{n \in \mathbb{Z}} (n\pi - \arcsin(0.8\gamma), n\pi + \arcsin(0.8\gamma)).$$

For each  $1 \leq i < e(1) - 1$ ,  $y^i$  and  $y^{i+1}$  must lie in the same component of the last set, since

$$y^{i+1} - y^i = \frac{2\pi}{b} \leq \frac{\pi}{3} < \pi - 2 \arcsin(0.8\gamma).$$

Therefore, there exists  $n_0 \in \mathbb{Z}$  such that

$$y^i - n_0\pi \in (-\arcsin(0.8\gamma), \arcsin(0.8\gamma)) \text{ for each } i \in \{1, 2, \dots, e(1) - 1\}.$$

Consequently,

$$(3.22) \quad e(1) - 2 = \frac{y^{e(1)-1} - y^1}{2\pi/b} < \frac{2 \arcsin(0.8\gamma)}{2\pi/b} \leq 0.4\gamma b,$$

where we used  $\arcsin t \leq \pi t/2$  for each  $t \in [0, 1]$ . If  $2 + 0.4\gamma b \leq \gamma b$ , then we are done. So assume the contrary. Then  $\gamma b < 10/3$  and hence  $e(1) - 2 < 4/3$ . Therefore  $e(1) \leq 3$ . If  $\gamma > 1/2$ , then  $e(1) < \gamma b$  holds. So assume  $\gamma \leq 1/2$ . Then

$$\frac{\arcsin(0.8\gamma)}{0.8\gamma} \leq \frac{\arcsin(0.5)}{0.5} = \frac{\pi}{3},$$

and hence (3.22) improves to the following  $e(1) - 2 < 4\gamma b/15$ . If  $2 + 4\gamma b/15 \leq \gamma b$  then we are done. So assume  $2 + 4\gamma b/15 > \gamma b$ . Then  $\gamma b < 30/11$  and hence  $e(1) - 2 < 4\gamma b/15 < 1$ . It follows that  $e(1) = 1$  or  $2$ . If  $\gamma b > 2$  then  $e(1) < \gamma b$ . So assume  $\gamma b \leq 2$ . To complete the proof we need to show  $e(1) = 1$ . By (3.5), it suffices to show

$$\left(\frac{2\gamma}{b-\gamma}\right)^2 + \left(\frac{2\gamma\Delta_{b,\gamma}}{1-\gamma^2}\right)^2 < 4\sin^2 \frac{\pi}{b}.$$

Since  $\gamma b \leq 2$ , we are reduced to show

$$(3.23) \quad \frac{16}{(b^2-2)^2} + \frac{16}{(b-2)^2} \left(\frac{\Delta_{b,2/b}}{1+2/b}\right)^2 < 4\sin^2 \frac{\pi}{b}.$$

In the case  $b = 6$ , by (3.1),  $\Delta_{6,1/3} \leq \max(0.99 + 1/3, 1 + 0.972/3) = 1.324$ , then an easy numerical calculation shows that the left hand side of (3.23) is less than the right hand side which is equal to 1. Assume now  $b \geq 7$ . Using  $\Delta_{b,2/b} \leq 1 + 2/b$ , we are further reduced to show

$$(3.24) \quad \frac{4b^2}{(b^2-2)^2} + \frac{4b^2}{(b-2)^2} < b^2 \sin^2 \frac{\pi}{b}.$$

Note that the left hand side is decreasing in  $b$  and the right hand side is increasing in  $b$ . Thus it suffices to verify this inequality in the case  $b = 7$ , which is an easy exercise.  $\square$

**3.2. The case  $b = 5$ .** We use  $\sin(\pi/5) = \sqrt{10 - 2\sqrt{5}}/4$ . By (3.6), for each  $1 \leq i < e(1)$ , since  $\gamma < 1$ ,

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4\sin^2 \frac{\pi}{5} - (4/4)^2} = (\sqrt{5} - 1)/2 > 0.6.$$

Moreover, by (3.8) if either  $k_i = k$  or  $k_{i+1} = k$ , then

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4\sin^2 \frac{\pi}{5} - (2/4)^2} = \sqrt{9 - \sqrt{5}}/2 > 1.$$

Thus

$$2 \geq |\sin(2\pi x_{e(1)}) - \sin(2\pi x_1)| > 1 + 0.6(e(1) - 2),$$

which implies  $e(1) \leq 3$ , since  $e(1)$  is an integer. If  $\gamma > 3/5$  then  $e(1) < \gamma b$ . Assume now  $\gamma \leq 3/5$ . Then by (3.6), for each  $1 \leq i < e(1)$ ,

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4 \sin^2 \frac{\pi}{5} - (3/5)^2} = \sqrt{(4.64 - \sqrt{5})/2} > 1.$$

Thus  $2 \geq 1.1 + (e(1) - 2)$  which implies  $e(1) \leq 2$ .

**3.3. The case  $b = 4$ .** We use  $\sin(\pi/4) = \sqrt{2}/2$ . By (3.6), for each  $1 \leq i < e(1)$ , since  $\gamma < 1$ ,

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4 \sin^2 \frac{\pi}{4} - (4/3)^2} = \frac{\sqrt{2}}{3}.$$

Moreover, by (3.8), if  $k_i = k$  or  $k_{i+1} = k$ , then

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4 \sin^2 \frac{\pi}{4} - (2/3)^2} = \frac{\sqrt{14}}{3}.$$

Thus

$$2 \geq |\sin(2\pi x_{e(1)}) - \sin(2\pi x_1)| \geq \frac{\sqrt{14}}{3} + (e(1) - 2) \frac{\sqrt{2}}{3},$$

which implies  $e(1) \leq 3$ . Therefore, either  $e(1) < \gamma b$  or  $\gamma \leq 3/4$ . Assume the latter. Then by (3.6), for each  $1 \leq i < e(1)$  we have

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4 \sin^2 \frac{\pi}{4} - 1} = 1.$$

Thus  $2 \geq \sqrt{14}/3 + (e(1) - 2)$ , which implies  $e(1) \leq 2$ .

**3.4. The case  $b = 3$ .** We use  $\sin(\pi/3) = \sqrt{3}/2$ . We claim that for each  $z \in [0, 1]$ ,  $E(1, z) \neq \{0, 1, 2\}^2$ , so that by Lemma 2.7,  $\sigma(1) \leq \sqrt{2} + 1$ . Otherwise, there exists  $z \in [0, 1]$  such that  $E(1, z) = \{0, 1, 2\}^2$ . Using the notation introduced above, for any  $1 \leq i < j \leq 3$ , as in (3.8), we have

$$|\sin(2\pi x_i) - \sin(2\pi x_j)| \geq \sqrt{4 \sin^2 \frac{\pi}{3} - \frac{4}{4}} = \sqrt{2},$$

which contradicts the fact

$$2 \geq \sin(2\pi x_3) - \sin(2\pi x_1) = |\sin(2\pi x_3) - \sin(2\pi x_2)| + |\sin(2\pi x_2) - \sin(2\pi x_1)|.$$

Assume  $\sigma(1) \geq \gamma b$ . Then  $\gamma < (1 + \sqrt{2})/3 < 0.81$ . Keep the notation  $x_j$ ,  $e(1)$  as above. By (3.6), for each  $1 \leq i < e(1)$  we have

$$|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{4 \sin^2 \frac{\pi}{3} - \frac{16\gamma^2}{(3-\gamma)^2}} > 0.9.$$

By (3.8), if  $k_i = k$  or  $k_{i+1} = k$ , then  $|\sin(2\pi x_i) - \sin(2\pi x_{i+1})| \geq \sqrt{2}$ . Thus  $2 \geq \sqrt{2} + (e(1) - 2) \cdot 0.9$  which implies that  $e(1) \leq 2$ .

4. PROOF OF THEOREM 2.3: THE CASE  $b = 3, 4, 5$ 

In this section, we shall prove Theorem 2.3 in the case  $b \in \{3, 4, 5\}$ . We shall need the following improvement of Lemma 3.2.

**Lemma 4.1.** *Let  $x^* \in [0, 1/2]$  and  $0 \leq k < l < b$  be such that  $(k, l) \in E(1, x^*)$ . Then for any  $\kappa \in (0, 1)$ , one of the following holds: either*

$$(4.1) \quad \left| \cos \frac{2\pi(x^* + k)}{b} - \cos \frac{2\pi(x^* + l)}{b} \right| \leq \frac{2\gamma\sqrt{1-\kappa^2}}{b} + \frac{2\gamma^2}{b(b-\gamma)},$$

or

$$(4.2) \quad \left| \sin \frac{2\pi(x^* + k)}{b} - \sin \frac{2\pi(x^* + l)}{b} \right| \leq 2\kappa\gamma + \frac{2\gamma^2}{1-\gamma}.$$

*Proof.* By Proposition 2.2 (1), there exist  $\mathbf{k} = \{k_n\}_{n=1}^\infty$  and  $\mathbf{l} = \{l_n\}_{n=1}^\infty$  in  $\mathcal{A}^{\mathbb{Z}^+}$  with  $k_1 = k$  and  $l_1 = l$  and such that the function

$$(4.3) \quad F(x) = -\frac{1}{2\pi} (S(x, \mathbf{k}) - S(x, \mathbf{l})),$$

has a multiple zero at  $x = x^*$ . Let

$$(4.4) \quad x_n = \frac{x + k_1 + bk_2 + \cdots + b^{n-1}k_n}{b^n}, \quad y_n = \frac{x + l_1 + bl_2 + \cdots + b^{n-1}l_n}{b^n},$$

and let

$$P_n(x) = \sin(2\pi x_n) - \sin(2\pi y_n), \quad Q_n(x) = \cos(2\pi x_n) - \cos(2\pi y_n).$$

Since  $F(x^*) = \sum_{n=1}^\infty \gamma^{n-1} P_n(x^*)$ , we have

$$|P_1(x^*)| \leq \gamma |P_2(x^*)| + \sum_{n=3}^\infty 2\gamma^{n-1} = \gamma |P_2(x^*)| + \frac{2\gamma^2}{1-\gamma}.$$

If  $|P_2(x^*)| \leq 2\kappa$ , then this implies that (4.2) holds. Assume  $|P_2(x^*)| > 2\kappa$ . Since  $P_2(x^*)^2 + Q_2(x^*)^2 \leq 4$ , we have  $|Q_2(x^*)| \leq 2\sqrt{1-\kappa^2}$ . Since  $0 = \frac{bF'(x^*)}{2\pi} = \sum_{n=1}^\infty (\gamma/b)^{n-1} Q_n(x^*)$ , we conclude

$$|Q_1(x^*)| \leq \frac{\gamma}{b} |Q_2(x^*)| + \sum_{n=3}^\infty 2 \left(\frac{\gamma}{b}\right)^{n-1} \leq \frac{2\gamma\sqrt{1-\kappa^2}}{b} + \frac{2\gamma^2}{b(b-\gamma)},$$

which is (4.1).  $\square$

**4.1. The case  $b = 5$ .** By Theorem 3.1 (ii), to complete the proof of Theorem 2.3 in the case  $b = 5$ , it suffices to prove the following.

**Theorem 4.1.** *Assume  $b = 5$  and  $e(1) = 2$ . Then  $\sigma(1) < 5\gamma$ .*

**Lemma 4.2.** *Assume  $\gamma \leq 2/5$ . Let  $0 \leq x^* \leq 1/2$  and  $0 \leq k < l < 5$  be such that  $(k, l) \in E(1, x^*)$ . Then either*

$$0 \leq x^* < 1/10 \text{ and } (k, l) = (2, 3),$$

or

$$2/5 < x^* \leq 1/2 \text{ and } (k, l) = (0, 4).$$

*Proof.* Put  $x^*(k) = (x^* + k)/5$ ,  $x^*(l) = (x^* + l)/5$ , and  $y^* = \pi(2x^* + k + l)/5$ . We shall use Lemma 3.2 and Lemma 4.1 to prove

$$(4.5) \quad |\sin y^*| < \sin \frac{\pi}{25}$$

which implies the statement.

By (3.4) and (3.5) in Lemma 3.2, we have

$$(4.6) \quad |\cos(2\pi x^*(k)) - \cos(2\pi x^*(l))| \leq \frac{2\gamma}{5-\gamma} = \frac{4}{23},$$

and

$$4 \sin^2 \frac{\pi(l-k)}{5} \leq \left( \frac{2\gamma}{1-\gamma} \right)^2 + \left( \frac{2\gamma}{5-\gamma} \right)^2 < \left( \frac{4}{3} \right)^2 + \left( \frac{4}{23} \right)^2 < 2.$$

The latter inequality implies that  $l - k \equiv \pm 1 \pmod{5}$ .

Let  $\kappa = \sqrt{2}/2$ . Let us show that the inequality (4.2) does not hold. Indeed, otherwise, we would have

$$|\sin(2\pi x^*(k)) - \sin(2\pi x^*(l))| \leq \gamma\sqrt{2} + \frac{2\gamma^2}{1-\gamma} < 1.1,$$

which together with (4.6) would imply that

$$\begin{aligned} 1.38 \dots &= 4 \sin^2 \frac{\pi}{5} = 4 \sin^2 \frac{\pi(l-k)}{5} \\ &= |\cos(2\pi x^*(k)) - \cos(2\pi x^*(l))|^2 + |\sin(2\pi x^*(k)) - \sin(2\pi x^*(l))|^2 \\ &< (4/23)^2 + 1.1^2 < 1.3, \end{aligned}$$

which is absurd.

Therefore the inequality (4.1) holds. It follows that

$$2 \sin \frac{\pi}{5} |\sin y^*| = |\cos(2\pi x^*(k)) - \cos(2\pi x^*(l))| \leq \frac{\gamma\sqrt{2}}{5} + \frac{2\gamma^2}{5(5-\gamma)} < 0.128,$$

and hence  $|\sin(y^*)| < 0.11 < \sin(\pi/25)$ .  $\square$

**Lemma 4.3.** *If  $\gamma \leq (\sqrt{5} + 1)/10$ , then  $e(1) = 1$ .*

*Proof.* Suppose  $(k, l) \in E(1, x)$ . Then by (3.5) in Lemma 3.2, we obtain

$$4 \sin^2 \frac{(l-k)\pi}{5} < \left( \frac{2\gamma}{5-\gamma} \right)^2 + \left( \frac{2\gamma}{1-\gamma} \right)^2 < 4 \sin^2 \frac{\pi}{5}$$

which implies that  $k = l$ .  $\square$

*Proof of Theorem 4.1.* If  $\gamma > 2/5$ , then  $\sigma(1) \leq e(1) = 2 < 5\gamma$ . Assume now  $\gamma \leq 2/5$  so that Lemma 4.2 applies. By Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that if  $(k, l) \in E(1, x; \varepsilon, \delta)$  for some  $x \in [0, 1/2]$ , then we have either  $x \in [0, 1/10]$ ,  $(k, l) = (2, 3)$  or  $x \in (2/5, 1/2]$ ,  $(k, l) = (0, 4)$ .

Let  $K = [1/10, 2/5] \cup [3/5, 9/10]$ . Then by Lemma 2.9,  $e(1, x; \varepsilon, \delta) = 1$  for all  $x \in K$  and  $e(1, x; \varepsilon, \delta) \leq 2$  for all  $x \in [0, 1]$ , so the condition (i) in Lemma 2.11 is satisfied (for  $q = 1$ ). Let us prove that the condition (ii) is satisfied. Let  $x \in [0, 1] \setminus K$  and  $0 \leq k < l < 5$  be such that  $(k, l) \in E(1, x; \varepsilon, \delta)$ . We need to check either  $x(k) \in K$  or  $x(l) \in K$ . Indeed, by symmetry (Lemma 2.9), it suffices to consider the case  $x \in [0, 1/2] \setminus K$ ; while for  $x \in [0, 1/10]$ , we have  $(k, l) = (2, 3)$  and  $x(3) \in K$  and for  $x \in (2/5, 1/2]$  we have  $(k, l) = (0, 4)$

and  $x(4) \in K$ . Having verified the conditions in Lemma 2.11, we conclude  $\sigma(1) \leq (\sqrt{5}+1)/2$ . By Lemma 4.3,  $\gamma > (\sqrt{5}+1)/10$  since we assume  $e(1) = 2$ . Thus  $\sigma(1) < 5\gamma$ .  $\square$

**4.2. The case  $b = 4$ .** By Theorem 3.1 (ii), to complete the proof of Theorem 2.3 in the case  $b = 4$ , it suffices to prove the following.

**Theorem 4.2.** *Assume  $b = 4$  and  $e(1) = 2$ . Then  $\sigma(1) < b\gamma$ .*

First we apply Lemma 3.2 and Lemma 4.1 to obtain the following estimate.

**Lemma 4.4.** *Assume  $\gamma \leq 1/2$ . Let  $0 \leq x^* \leq 1/2$  and let  $0 \leq k < l < 4$  be such that  $(k, l) \in E(1, x^*)$ . Then either*

$$x^* \in [3/8, 1/2] \text{ and } (k, l) \in \{(0, 3), (1, 2)\},$$

or

$$x^* \in [0, 1/8] \text{ and } (k, l) = (1, 3).$$

*Proof.* By Lemma 3.2,

$$(4.7) \quad \left| \cos \frac{2\pi(x^* + k)}{4} - \cos \frac{2\pi(x^* + l)}{4} \right| \leq \frac{2\gamma}{4-\gamma} \leq \frac{2}{7}.$$

Let us apply Lemma 4.1 with  $\kappa = 1/3$ . We claim that (4.2) does not hold. Indeed, otherwise,

$$\left| \sin \frac{2\pi(x^* + k)}{4} - \sin \frac{2\pi(x^* + l)}{4} \right| \leq \frac{4}{3},$$

which together with (4.7) would imply that

$$\begin{aligned} 2 &\leq 4 \sin^2 \frac{\pi(l-k)}{4} \\ &= \left( \cos \frac{2\pi(x^* + k)}{4} - \cos \frac{2\pi(x^* + l)}{4} \right)^2 + \left( \sin \frac{2\pi(x^* + k)}{4} - \sin \frac{2\pi(x^* + l)}{4} \right)^2 \\ &\leq \left( \frac{2}{7} \right)^2 + \left( \frac{4}{3} \right)^2 < 2, \end{aligned}$$

which is absurd. Therefore, the inequality (4.1) holds with  $\kappa = 1/3$ , which implies that

$$2 \left| \sin \frac{\pi(l-k)}{4} \right| \left| \sin \left( \frac{2\pi x^*}{4} + \frac{\pi(k+l)}{4} \right) \right| \leq \frac{\sqrt{2}}{6} + \frac{1}{28}.$$

Consequently,

$$\left| \sin \left( \frac{2\pi x^*}{4} + \frac{\pi(k+l)}{4} \right) \right| \leq \frac{1}{6} + \frac{1}{28\sqrt{2}} < \sin \frac{\pi}{16}.$$

Since  $2\pi x^*/4 \in [0, \pi/4]$  the lemma follows.  $\square$

A bit more careful analysis shows that the second alternative in the lemma above never happens.

**Lemma 4.5.** *Assume  $\gamma \leq 1/2$ . Then for any  $x^* \in [0, 1/2]$ ,  $(1, 3) \notin E(1, x^*)$ .*



*Proof.* We shall prove this lemma by contradiction. Assume  $(1, 3) \in E(1, x^*)$ . Then there exists  $\mathbf{k} = \{k_n\}_{n=1}^\infty$  and  $\mathbf{l} = \{l_n\}_{n=1}^\infty$  with  $k_1 = 1$  and  $l_1 = 3$  and such that the function

$$F(x) = -\frac{1}{2\pi} (S(x, \mathbf{k}) - S(x, \mathbf{l}))$$

has a multiple zero at  $x^*$ . Let

$$x_n = \frac{x + k_1 + 4k_2 + \cdots + 4^{n-1}k_n}{4^n}, \quad y_n = \frac{x + l_1 + 4l_2 + \cdots + 4^{n-1}l_n}{4^n},$$

and let

$$P_n(x) = \sin(2\pi x_n) - \sin(2\pi y_n), \quad Q_n(x) = \cos(2\pi x_n) - \cos(2\pi y_n).$$

Then  $F(x) = \sum_{n=1}^\infty \gamma^{n-1} P_n(x)$ . Since  $F(x^*) = 0$ , this gives us

$$|P_1(x^*) + \gamma P_2(x^*)| \leq \sum_{n=3}^\infty \gamma^{n-1} |P_n(x^*)| \leq 1.$$

Note that

$$P_2(x) \geq -\cos \frac{\pi(1+x)}{4} - \cos \frac{\pi(1-x)}{4} = -2 \cos \frac{\pi}{4} \cos \frac{\pi x}{4} \geq -\sqrt{2}.$$

Therefore

$$P_1(x^*) \leq 1 - \gamma P_2(x^*) \leq 1 + \frac{\sqrt{2}}{2}.$$

As in the previous lemma,  $|Q_1(x^*)| \leq 2/7$ . Since  $P_1(x^*) > 0$ , we have

$$P_1(x^*)^2 + Q_1(x^*)^2 \leq (2/7)^2 + (1 + \sqrt{2}/2)^2 < 4.$$

However, the left hand of the inequality is equal to 4, a contradiction!  $\square$

**Lemma 4.6.** *If  $\gamma \leq (\sqrt{5} + 1)/8$  then  $e(1) = 1$ .*

*Proof.* For  $x \in [0, 1]$  and  $0 \leq k < l < 4$ , if  $(k, l) \in E(1, x)$ , then by the inequality (3.5) in Lemma 3.2, we have

$$4 \sin^2 \frac{\pi(l-k)}{4} < \left( \frac{2\gamma}{4-\gamma} \right)^2 + \left( \frac{2\gamma}{1-\gamma} \right)^2 < 2,$$

which implies that  $l = k$ .  $\square$

*Proof of Theorem 4.2.* If  $\gamma > 1/2$ , then  $\sigma(1) \leq e(1) = 2 < 4\gamma$ . So assume  $\gamma \leq 1/2$ . By Lemmas 4.4 and 4.5 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that if  $x \in [0, 1/2]$  and  $0 \leq k < l < 4$  are such that  $(k, l) \in E(1, x; \varepsilon, \delta)$  then  $3/8 < x \leq 1/2$  and  $(k, l) \in \{(0, 3), (1, 2)\}$ . Note that for  $x \in (3/8, 1/2]$ , we have  $x(0), x(1) \in [0, 3/8]$ . By Lemma 2.9, it is then easy to check that the conditions of Lemma 2.11 are satisfied for  $K = [0, 3/8] \cup [5/8, 1]$  and  $q = 1$ . Thus  $\sigma(1) \leq (\sqrt{5} + 1)/2$ . On the other hand, since we assume  $e(1) = 2$ , by Lemma 4.6, we have  $\gamma > (\sqrt{5} + 1)/8$ . This proves that  $\sigma(1) < 4\gamma$ .  $\square$

**4.3. The case  $b = 3$ .** In this subsection, we shall prove the following theorem, which together with Theorem 3.1, implies Theorem 2.3 in the case  $b = 3$ .

**Theorem 4.3.** *Assume  $b = 3$  and  $e(1) = 2$ . Then  $\sigma(1) < b\gamma$ .*

**Lemma 4.7.** *Assume  $\gamma \leq 2/3$ . Let  $0 \leq x^* \leq 1/2$  and let  $0 \leq k < l < 3$  be such that  $(k, l) \in E(1, x)$ . Then either*

$$x \in [0, 1/6] \text{ and } (i, j) = (1, 2);$$

or

$$x \in (1/3, 1/2] \text{ and } (i, j) = (0, 2).$$

*Proof.* If  $(k, l) \in E(1, x^*)$ , then by (3.4),

$$\left| \cos \frac{2\pi(x^* + k)}{3} - \cos \frac{2\pi(x^* + l)}{3} \right| \leq \frac{2\gamma}{3 - \gamma} \leq \frac{4}{7},$$

which implies that

$$\left| \sin \left( \frac{2\pi x^*}{3} + \frac{\pi(k + l)}{3} \right) \right| \leq \frac{4}{7 \cdot 2 |\sin \frac{\pi(l - k)}{3}|} = \frac{4}{7\sqrt{3}} < \sin \frac{\pi}{9}.$$

The statement follows.  $\square$

**Lemma 4.8.** *Assume  $\gamma \leq (\sqrt{5} + 1)/6$ . Let  $0 \leq x \leq 1/2$  and let  $0 \leq k < l < 3$  be such that  $(k, l) \in E(1, x)$ . Then either*

$$x \in [0, 1/8] \text{ and } (k, l) = (1, 2);$$

or

$$x \in [3/8, 1/2] \text{ and } (k, l) = (0, 2).$$

*Proof.* Under current assumption, again by (3.4), we have

$$\left| \cos \frac{2\pi(x^* + k)}{3} - \cos \frac{2\pi(x^* + l)}{3} \right| \leq \frac{2\gamma}{3 - \gamma} \leq 0.44,$$

hence

$$\left| \sin \left( \frac{2\pi x^*}{3} + \frac{\pi(k + l)}{3} \right) \right| \leq \frac{0.44}{2 |\sin \frac{\pi(l - k)}{3}|} = \frac{0.44}{\sqrt{3}} < \sin \frac{\pi}{12}.$$

Thus the statement holds.  $\square$

**Lemma 4.9.** *If  $3\gamma \leq \sqrt{2}$ , then  $e(1) = 1$ .*

*Proof.* By (3.2) and (3.5), if  $(k, l) \in E(1, x^*)$  for some  $0 \leq k, l < 3$  and  $x^* \in \mathbb{R}$ , then

$$4 \sin^2 \frac{\pi(l - k)}{3} \leq \left( \frac{2\gamma}{3 - \gamma} \right)^2 + \left( \frac{2\gamma(1 + 0.71\gamma)}{1 - \gamma^2} \right)^2.$$

Since  $\gamma \leq \sqrt{2}/3$ , the right hand side is less than 3, which implies that  $k = l$ . This proves that  $e(1) = 1$ .  $\square$

*Proof of Theorem 4.3.* If  $\gamma > 2/3$ , then  $\sigma(1) \leq e(1) = 2 < 3\gamma$ . So assume  $\gamma \leq 2/3$ . By Lemma 4.7 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that if  $x \in [0, 1/2]$ ,  $0 \leq k < l < 3$  are such that  $(k, l) \in E(1, x; \varepsilon, \delta)$ , then either  $x \in [0, 1/6]$ ,  $(k, l) = (1, 2)$  or  $x \in (1/3, 1/2]$ ,  $(k, l) = (0, 2)$ . Note that for  $x \in [0, 1/6]$ , we have  $x(2) \in [2/3, 5/6]$  and for  $x \in (1/3, 1/2]$ , we have

$x(0) \in [1/6, 1/3]$ . By Lemma 2.9, it follows that the conditions in Lemma 2.11 are satisfied with  $K = [1/6, 1/3] \cup [2/3, 5/6]$  and  $q = 1$ . Thus  $\sigma(1) \leq (\sqrt{5}+1)/2$ .

If  $\gamma > (\sqrt{5}+1)/6$ , then  $\sigma(1) < 3\gamma$ . Assume  $\gamma \leq (\sqrt{5}+1)/6$ . Then by Lemma 4.8 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that if  $x \in [0, 1/2]$ ,  $0 \leq k < l < 3$  are such that  $(k, l) \in E(1, x; \varepsilon, \delta)$ , then either  $x \in [0, 1/8]$ ,  $(k, l) = (1, 2)$  or  $x \in (3/8, 1/2]$ ,  $(k, l) = (0, 2)$ . Note that for  $x \in [0, 1/8]$ , we have  $x(1) \in [1/8, 3/8]$ ,  $x(2) \in [5/8, 7/8]$  and for  $x \in [3/8, 1/2]$  we have  $x(0) \in [1/8, 3/8]$  and  $x(2) \in [5/8, 7/8]$ . By Lemma 2.9, it follows that the conditions in Lemma 2.10 are satisfied with  $K = [0, 3/8] \cup [5/8, 1)$  and  $q = 1$ . Thus  $\sigma(1) \leq \sqrt{2}$ . Since we assume  $e(1) = 2$ , by Lemma 4.9,  $\gamma > \sqrt{2}/3$ . This proves  $\sigma(1) < 3\gamma$ .  $\square$

## 5. THE CASE $b = 2$

This section is devoted to the proof of Theorem 2.3 in the case  $b = 2$ . The proof is structurally similar to the cases  $b = 3, 4, 5$  which we discussed above, but it is more involved and consists of several steps.

We shall use the following notation. For any  $\mathbf{k}, \mathbf{l} \in \mathcal{A}^q$  and  $x_* \in \mathbb{R}$ , we write  $\mathbf{k} \sim_{x_*} \mathbf{l}$  if  $(\mathbf{k}, \mathbf{l}) \in E(q, x_*)$ . In order to show  $\mathbf{k} \not\sim_{x_*} \mathbf{l}$ , by Proposition 2.2 (1), it suffices to show that any  $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$ , the function  $S(x, \mathbf{k}\mathbf{u}) - S(x, \mathbf{l}\mathbf{v})$  does not have a multiple zero at  $x_*$ .

**5.1. Step 1. When  $\gamma > \sqrt{2}/2$ .** In this step, we shall prove

**Proposition 5.1.** (1) For any  $\gamma \in (1/2, 1)$ ,  $\sigma(1) \leq (\sqrt{5}+1)/2$ .  
 (2) For any  $\gamma \in (1/2, (\sqrt{5}+1)/4]$ ,  $\sigma(1) \leq \sqrt{2}$ .

An immediate corollary is the following:

**Corollary 5.2.** Assume  $b = 2$  and  $\gamma > \sqrt{2}/2$ . Then  $\sigma(1) < 2\gamma$ .

The proof of this proposition relies on the following estimates.

**Lemma 5.3.** (i) For any  $x \in [0, 1/3]$ , we have

$$(00) \not\sim_x (10), (00) \not\sim_x (11), \text{ and } (01) \not\sim_x (10).$$

(ii) If either  $x \in [0, 1/4]$ , or  $x \in [0, 1/3]$  and  $\gamma \leq (\sqrt{5}+1)/4$ , then  $(01) \not\sim_x (11)$ .

*Proof.* Let  $\mathbf{u} = \{u_n\}_{n=1}^\infty$  and  $\mathbf{v} = \{v_n\}_{n=1}^\infty$  be elements in  $\mathcal{A}^{\mathbb{Z}^+}$  and let

$$F(x) = -\frac{1}{2\pi} (S(x, \mathbf{u}) - S(x, \mathbf{v})).$$

As before, given  $x \in \mathbb{R}$ , we write

$$x_n = \frac{x}{2^n} + \frac{u_1}{2^n} + \cdots + \frac{u_n}{2^n}, \quad y_n = \frac{x}{2^n} + \frac{v_1}{2^n} + \cdots + \frac{v_n}{2^n}$$

and write  $Q_n = \cos(2\pi x_n) - \cos(2\pi y_n)$ . Then  $|Q_n| \leq 2$  for all  $n$  and

$$G(x) := \frac{F'(x)}{\pi} = \sum_{n=1}^{\infty} \left(\frac{\gamma}{2}\right)^{n-1} Q_n.$$

In the following, we assume  $u_1 = 0$  and  $v_1 = 1$ , so that  $Q_1 = 2\cos(\pi x)$ .

(i). Assume first  $u_2 = 0$ . Then

$$Q_2 = \cos \frac{\pi x}{2} + (-1)^{v_2} \sin \frac{\pi x}{2} \geq \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} > 0$$

for any  $x \in [0, 1/2]$ . Thus

$$G(x) \geq Q_1 - \sum_{n=3}^{\infty} (\gamma/2)^{n-1} |Q_n| > 2 \cos(\pi x) - 1 \geq 0.$$

whenever  $0 \leq x \leq 1/3$ . This proves that for any  $x \in [0, 1/3]$ ,  $(00) \not\prec_x (10)$  and  $(00) \not\prec_x (11)$ .

To prove  $(01) \not\prec_x (10)$  for  $x \in [0, 1/3]$ , let  $u_2 = 1$  and  $v_2 = 0$ . Then

$$Q_2 = \sin(\pi x/2) - \cos(\pi x/2),$$

and

$$Q_3 = \pm \sin(\pi x/4) \pm \cos(\pi(1+x)/4) \geq -\sin(\pi/12) - \cos(\pi/4) > -1.$$

Thus

$$\begin{aligned} G(x) &\geq Q_1 + \frac{\gamma}{2} Q_2 + \frac{\gamma^2}{4} Q_3 + \sum_{n=3}^{\infty} \frac{\gamma^n}{2^n} Q_{n+1} \\ &\geq g(x) - \frac{\gamma^2}{4} - \frac{\gamma^3}{4-2\gamma} \geq g(x) - \frac{3}{4}, \end{aligned}$$

where

$$g(x) = 2 \cos(\pi x) + \frac{1}{2} \left( \sin \frac{\pi x}{2} - \cos \frac{\pi x}{2} \right).$$

Since

$$g''(x) = -2\pi^2 \cos(\pi x) - \pi^2/8(\sin(\pi x/2) - \cos(\pi x/2)) < 0$$

holds for all  $x \in [0, 1/3]$ , we have

$$\min_{x \in [0, 1/3]} g(x) = \min(g(0), g(1/3)) = \min\left(\frac{3}{2}, \frac{5-\sqrt{3}}{4}\right) > \frac{3}{4}.$$

Therefore  $G > 0$ .

(ii) Now let us assume  $u_2 = v_2 = 1$ . Then

$$Q_2 = -\sin \frac{\pi x}{2} - \cos \frac{\pi x}{2},$$

and

$$Q_3 = \pm \sin \frac{\pi x}{4} \pm \sin \frac{\pi(1+x)}{4} \geq -2 \sin \frac{\pi(1+2x)}{8} \cos \frac{\pi}{8}.$$

Thus

$$G(x) \geq Q_1 + \frac{\gamma}{2} Q_2 + \frac{\gamma^2}{4} Q_3 - 2 \sum_{n=4}^{\infty} \left(\frac{\gamma}{2}\right)^{n-1} \geq h_\gamma(x) - \frac{\gamma^3}{4-2\gamma},$$

where

$$h_\gamma(x) = 2 \cos(\pi x) - \frac{\gamma}{2} \left( \cos \frac{\pi x}{2} + \sin \frac{\pi x}{2} \right) - \frac{\gamma^2}{2} \cos \frac{\pi}{8} \sin \frac{\pi(1+2x)}{8}.$$

Assume first  $x \in [0, 1/4]$ . Since  $h_\gamma(x)$  is decreasing in both  $x$  and  $\gamma$ , we have

$$h_\gamma(x) \geq h_1(1/4) = \sqrt{2} - \frac{\sqrt{2}}{2} \sin \frac{3\pi}{8} - \frac{1}{2} \cos \frac{\pi}{8} \sin \frac{3\pi}{16} > \frac{1}{2} > \frac{\gamma^3}{4-2\gamma},$$

hence  $G(x) > 0$ . This proves that  $(01) \not\prec_x (11)$  for  $x \in [0, 1/4]$ .

Assume now  $\gamma \leq (\sqrt{5} + 1)/4 =: \gamma_0$ . Then by numerical calculation, we have

$$h_{\gamma_0}(1/3) = 1 - \frac{\gamma_0}{2} \frac{\sqrt{3} + 1}{2} - \frac{\gamma_0^2}{2} \sin \frac{\pi}{8} \sin \frac{5\pi}{24} > \frac{\gamma_0^3}{4 - 2\gamma_0}.$$

Thus for each  $0 \leq x \leq 1/3$ , we have

$$G(x) \geq h_\gamma(x) - \gamma^3/(4 - 2\gamma) \geq h_{\gamma_0}(1/3) - \gamma_0^3/(4 - 2\gamma_0) > 0.$$

This proves that  $(01) \not\prec_x (11)$  for all  $x \in [0, 1/3]$ .  $\square$

*Proof of Proposition 5.1.* By Lemma 5.3 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that  $e(1, x; \varepsilon, \delta) = 1$  for  $x \in [0, 1/4]$ . By Lemma 2.9, this also holds for  $x \in [3/4, 1]$ . For  $x \in [1/4, 1/2]$ ,  $x(0) \in [0, 1/4]$  while for  $x \in [1/2, 3/4]$  we have  $x(1) \in [3/4, 1]$ . By Lemma 2.11, we obtain  $\sigma(1) \leq (\sqrt{5} + 1)/2$ .

Assume that  $\gamma \leq (\sqrt{5} + 1)/4$ . Then Lemma 5.3 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that  $e(1, x; \varepsilon, \delta) = 1$  for all  $x \in [0, 1/3]$  and by Lemma 2.9, this also holds for  $x \in [2/3, 1]$ . For  $x \in (1/3, 2/3)$ , both  $x(0)$  and  $x(1)$  belong to  $[0, 1/3] \cup [2/3, 1]$ . By Lemma 2.10, it follows that  $\sigma(1) \leq \sqrt{2}$ .  $\square$

**5.2. Step 2: When  $\gamma > 0.64$ .** In this section we shall prove the following

**Proposition 5.4.** *Assume  $\gamma \leq \sqrt{2}/2$ . Then  $\sigma(2) \leq 1.61$ .*

As an immediate corollary of this proposition and Corollary 5.2, we have

**Corollary 5.5.** *If  $\gamma > 0.64$  then either  $\sigma(1) < 2\gamma$  or  $\sigma(2) < (2\gamma)^2$ .*

The proof of Proposition 5.4 relies on the following estimates.

**Lemma 5.6.** *Assume  $\gamma \leq \sqrt{2}/2$ . Then*

- (i) *For any  $x \in [0, 1/2]$ ,  $(00) \not\prec_x (10)$ ;*
- (ii) *For any  $x \in [0, 2/5]$ ,  $(01) \not\prec_x (10)$  and  $(00) \not\prec_x (11)$ ;*
- (iii) *For any  $x \in [0, 1/2]$ ,  $(00) \not\prec_x (11)$ ;*
- (iv) *For any  $x \in [0, 2/5]$ ,  $(01) \not\prec_x (11)$ ;*
- (v) *For any  $x \in [0, 2/5]$ ,  $0 \not\prec_x 1$ ;*
- (vi) *For  $x \in [1/5, 1/2]$ ,  $(10) \not\prec_x (11)$ .*

*Proof.* Let  $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$  and let  $F(x) = -(2\pi)^{-1}(S(x, \mathbf{u}) - S(x, \mathbf{v}))$ ,  $G(x) = F'(x)/\pi$ . For given  $x$ , we shall use the notations  $x_n, y_n, Q_n$  as in Lemma 5.3 and let  $P_n = \sin(2\pi x_n) - \sin(2\pi y_n)$ .

(i) Assume  $(u_1, u_2) = (0, 0)$  and  $(v_1, v_2) = (1, 0)$ . Then  $Q_1 = 2\cos(\pi x)$  and  $Q_2 = \cos \frac{\pi x}{2} + \sin \frac{\pi x}{2}$ . Thus

$$G(x) = Q_1 + \frac{\gamma}{2}Q_2 + \cdots \geq f(x) - \frac{\gamma^2}{2 - \gamma},$$

where

$$f(x) = 2\cos(\pi x) + \frac{\gamma}{2} \left( \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2} \right).$$

On the interval  $x \in [0, 1/2]$ , we have

$$f''(x) = -2\pi^2 \sin(\pi x) - \frac{\pi^2 \gamma}{8} \left( \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2} \right) < 0.$$

Thus

$$\min_{x \in [0, 1/2]} f(x) = \min(f(0), f(1/2)) = \min\left(2 + \frac{\gamma}{2}, \frac{\gamma}{\sqrt{2}}\right) = \frac{\gamma}{\sqrt{2}} > \frac{\gamma^2}{2 - \gamma},$$

hence  $G(x) > 0$ . Therefore, for any  $x \in [0, 1/2]$ ,  $(00) \not\prec_x (11)$ .

(ii) Assume either  $(u_1, u_2) = (0, 1)$  and  $(v_1, v_2) = (1, 0)$ ; or  $(u_1, u_2) = (0, 0)$  and  $(v_1, v_2) = (1, 1)$ . Then

$$Q_2 = \pm \left( \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} \right) \geq \sin \frac{\pi x}{2} - \cos \frac{\pi x}{2}.$$

Thus

$$\begin{aligned} G(x) &\geq 2 \cos(\pi x) - \frac{\gamma}{2} \left( \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} \right) - \frac{\gamma^2}{2 - \gamma} \\ &\geq 2 \cos(\pi x) - \frac{1}{2} \sin \frac{\pi(1 - 2x)}{4} - \frac{4 + \sqrt{2}}{14} := g(x). \end{aligned}$$

For  $x \in [0, 2/5]$ , we have

$$g''(x) = -2\pi^2 \cos(\pi x) + \frac{\pi^2}{8} \sin \frac{\pi(1 - 2x)}{4} < 0.$$

Thus  $g(x) \geq \min(g(0), g(2/5))$ . Clearly,  $g(0) > 0$  and  $g(2/5) > 0$ . It follows that  $G(x) > 0$  for all  $x \in [0, 2/5]$ . This proves that  $(01) \not\prec_x (10)$  and  $(00) \not\prec_x (11)$  for  $x \in [0, 2/5]$ .

(iii) Assume  $(u_1, u_2) = (0, 0)$  and  $(v_1, v_2) = (1, 1)$ . By (ii), it suffices to show that  $F(x) > 0$  for any  $x \in [2/5, 1/2]$ . Note that  $P_1 = 2 \sin(\pi x)$ ,

$$P_2 = \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2} = \sqrt{2} \sin \left( \frac{\pi x}{2} + \frac{\pi}{4} \right),$$

and

$$P_3 \geq -\sin \frac{\pi x}{4} - \sin \frac{\pi(1 - x)}{4} = -2 \sin \frac{\pi}{8} \cos \frac{\pi(1 - 2x)}{8} \geq -2 \sin \frac{\pi}{8},$$

so

$$\frac{F(x)}{\gamma} = \sum_{n=1}^{\infty} \gamma^{n-2} P_n \geq \frac{2 \sin(\pi x)}{\gamma} + \sqrt{2} \sin \left( \frac{\pi x}{2} + \frac{\pi}{4} \right) - 2\gamma \sin \frac{\pi}{8} - \frac{2\gamma^2}{1 - \gamma}$$

The right hand side is increasing in  $x \in [2/5, 1/2]$  and decreasing in  $\gamma$ . Thus for  $x \in [2/5, 1/2]$ ,

$$\frac{F(x)}{\gamma} \geq 2\sqrt{2} \sin \frac{2\pi}{5} + \sqrt{2} \sin \frac{9\pi}{20} - \sqrt{2} \sin \frac{\pi}{8} - 2 - \sqrt{2} > 0.$$

(iv) Assume  $(u_1, u_2) = (0, 1)$ ,  $(v_1, v_2) = (1, 1)$ . By Lemma 5.3, we only need to show that  $(01) \not\prec_x (11)$  for  $x \in [1/3, 2/5]$ . Then

$$P_2 = \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} \text{ and } Q_2 = -\sin \frac{\pi x}{2} - \cos \frac{\pi x}{2}.$$

Put

$$(5.1) \quad p(x) = \frac{P_1}{\gamma} + P_2 = \frac{2 \sin(\pi x)}{\gamma} + \left( \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} \right),$$

$$(5.2) \quad q(x) = Q_1 + \frac{\gamma}{2} Q_2 = 2 \cos(\pi x) - \frac{\gamma}{2} \left( \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2} \right).$$

Since  $q(x)$  is decreasing in  $[0, 1/2]$ , we have

$$(5.3) \quad q(x) \geq q(2/5) > \frac{\sqrt{5}-1}{2} - \frac{\gamma}{\sqrt{2}} \geq \frac{\sqrt{5}-2}{2}.$$

for each  $0 \leq x \leq 2/5$ . Since  $p' = \pi q/\gamma > 0$ , for each  $x \in [1/3, 2/5]$ , we have

$$(5.4) \quad p(x) \geq p(1/3) \geq \frac{\sqrt{3}}{\gamma} + \frac{\sqrt{3}-1}{2} \geq \sqrt{6} + \frac{\sqrt{3}-1}{2} > 2.81.$$

**Case 1.** Assume also  $v_3 = 0$ . Then

$$Q_3 = \pm \sin \frac{\pi x}{4} + \cos \frac{\pi(1-x)}{4} \geq \cos \frac{\pi(1-x)}{4} - \sin \frac{\pi x}{4}.$$

Since the right hand side is decreasing in  $[0, 1/2]$ , we obtain

$$Q_3 \geq \cos \frac{\pi}{8} - \sin \frac{\pi}{8} > \frac{1}{2},$$

and

$$G(x) \geq Q_1 + \frac{\gamma}{2}Q_2 + \frac{\gamma^2}{4}Q_3 - \frac{\gamma^3}{4-2\gamma} > q(x) + \frac{\gamma^2}{8} - \frac{\gamma^3}{4-2\gamma}.$$

Thus

$$\frac{G(x)}{\gamma^2} \geq \frac{\sqrt{5}-2}{2\gamma^2} + \frac{1}{8} - \frac{\gamma}{4-2\gamma} \geq \sqrt{5}-2 + \frac{1}{8} - \frac{1}{4\sqrt{2}-2} > 0.$$

**Case 2.** Assume  $u_3 = v_3 = 1$ . Then

$$P_3 = \cos \frac{\pi(1+x)}{4} - \cos \frac{\pi x}{4} = -2 \sin \frac{\pi(1+2x)}{8} \sin \frac{\pi}{8} > -\sqrt{2} \sin \frac{\pi}{8}$$

and

$$Q_3 = \sin \frac{\pi x}{4} - \sin \frac{\pi(1+x)}{4} = -2 \sin \frac{\pi}{8} \cos \frac{\pi(1+2x)}{8} > -\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2}.$$

**Subcase 2.1.**  $u_4 = 1$  and  $v_4 = 0$ . Then

$$Q_4 = \cos \frac{\pi(2-x)}{8} + \cos \frac{\pi(1-x)}{8} > \sqrt{2} > \frac{2\gamma}{2-\gamma}.$$

Therefore, for  $x \in [0, 2/5]$ ,

$$\begin{aligned} G(x) &= Q_1 + \frac{\gamma}{2}Q_2 + \frac{\gamma^2}{4}Q_3 + \frac{\gamma^3}{8}Q_4 - \frac{\gamma^4}{8-4\gamma} \\ &\geq q(x) + \frac{\gamma^2}{4}Q_3 + \frac{\gamma^3}{8} \left( Q_4 - \frac{2\gamma}{2-\gamma} \right) \\ &> q(x) - \frac{\sqrt{2}}{16} > 0, \end{aligned}$$

where the last inequality follows from (5.3).

**Subcase 2.2.**  $u_4 = 0$ . Then

$$P_4 \geq \sin \frac{\pi(2-x)}{8} - \sin \frac{\pi(1-x)}{8} > 0.$$

Thus

$$\begin{aligned} \frac{F(x)}{\gamma} &\geq \frac{P_1}{\gamma} + P_2 + \gamma P_3 + \gamma^2 P_4 - \frac{2\gamma^3}{1-\gamma} > p(x) + \gamma P_3 - \frac{2\gamma^3}{1-\gamma} \\ &\geq p(x) - \sqrt{2}\gamma \sin \frac{\pi}{8} - \frac{2\gamma^3}{1-\gamma} \geq p(x) - \sin \frac{\pi}{8} - \sqrt{2} - 1 > 0.01, \end{aligned}$$

where the last inequality follows from (5.4).

**Subcase 2.3.**  $u_4 = v_4 = 1$ . Then

$$P_4 = \sin \frac{\pi(1-x)}{8} - \sin \frac{\pi(2-x)}{8} = -2 \sin \frac{\pi}{16} \cos \frac{\pi(3-2x)}{16} > -2 \sin \frac{\pi}{16};$$

and

$$P_5 \geq -\sin \frac{\pi(2-x)}{16} - \sin \frac{\pi(1-x)}{16} = -2 \sin \frac{\pi(3-2x)}{32} \cos \frac{\pi}{32} > -2 \sin \frac{3\pi}{32}.$$

Therefore,

$$\begin{aligned} \frac{F(x)}{\gamma} &\geq p(x) + \gamma P_3 + \gamma^2 P_4 + \gamma^3 P_5 - \frac{2\gamma^4}{1-\gamma} \\ &\geq p(x) - \gamma\sqrt{2} \sin \frac{\pi}{8} - 2\gamma^2 \sin \frac{\pi}{16} - 2\gamma^3 \sin \frac{3\pi}{32} - \frac{2\gamma^4}{1-\gamma} \\ &\geq p(x) - \sin \frac{\pi}{8} - \sin \frac{\pi}{16} - \frac{1}{\sqrt{2}} \sin \frac{3\pi}{32} - 1 - \frac{\sqrt{2}}{2} > 0, \end{aligned}$$

where the last inequality follows from (5.4).

**Case 3.** Assume  $u_3 = 0$  and  $v_3 = 1$ . Then for  $x \in [1/3, 2/5]$ ,

$$P_3 = \cos \frac{\pi x}{4} + \cos \frac{\pi(1+x)}{4} = 2 \cos \frac{\pi}{8} \cos \frac{\pi(1+2x)}{8} > \sqrt{2} \cos \frac{\pi}{8}.$$

Thus

$$\begin{aligned} \frac{1}{\gamma^2} F(x) &\geq \frac{1}{\gamma^2} P_1 + \frac{P_2}{\gamma} + P_3 - \frac{2\gamma}{1-\gamma} > \frac{p(x)}{\gamma} + \sqrt{2} \cos \frac{\pi}{8} - \frac{2\gamma}{1-\gamma} \\ &\geq \sqrt{2} \left( p(x) + \cos \frac{\pi}{8} - 2 - \sqrt{2} \right) > 0, \end{aligned}$$

where the last inequality follows from (5.4).

(v) It follows from (i) (ii) and (iv).

(vi) It suffices to show that  $0 \not\sim_{x(1)} 1$  for  $x \in [1/5, 1/2]$ . But  $x(1) \in [3/5, 1]$ , so the statement follows from (v) and Lemma 2.9.  $\square$

Summarizing the estimates given by Lemma 5.3 and Lemma 5.6, we have

**Lemma 5.7.** Assume  $\gamma \leq \sqrt{2}/2$ . Then

- (1) For  $x \in [1/5, 2/5]$ ,  $e(2, x) = 1$ ;
- (2) For  $x \in [0, 1/5)$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(10, 11)$  and  $(11, 10)$ ;
- (3) For  $x \in (2/5, 1/2]$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(01, 10)$ ,  $(01, 11)$ ,  $(10, 01)$  and  $(11, 01)$ .

*Proof.* By Lemma 5.3,  $(00) \not\sim_x (01)$  for all  $x \in [0, 1/2]$ , since  $0 \not\sim_{x(0)} 1$ . By Lemma 5.6 (i) and (iii), we have  $(00) \not\sim_x (10)$  and  $(00) \not\sim_x (11)$  for all  $x \in [0, 1/2]$ . For  $x \in [2/5, 1/2]$ , by Lemma 5.6 (vi), we also have  $(10) \not\sim_x (11)$ . Thus (3) holds. For  $x \in [0, 2/5]$ , by Lemma 5.6 (ii) and (iv),  $(01) \not\sim_x (10)$ ,



$(01) \not\sim_x (11)$ , so the only possible non-trivial pairs in  $E(2, x)$  are  $(10, 11)$  and  $(11, 10)$ . So (2) holds. If  $x \in [1/5, 2/5]$ , then  $(10) \not\sim_x (11)$  by Lemma 5.6 (vi). Therefore (1) holds.  $\square$

*Proof of Proposition 5.4.* We shall apply Lemma 2.12. Let  $K_0 = [1/5, 2/5] \cup [3/5, 4/5]$ ,  $K_1 = [0, 1/5] \cup (4/5, 1)$  and  $K_2 = (2/5, 3/5)$ . By Lemma 5.7, Proposition 2.2 (3) and Lemma 2.9, the conditions in Lemma 2.12 are satisfied with  $q = 2$  and with suitable choice of  $(\varepsilon, \delta)$ . Indeed,

- For  $x \in [1/5, 2/5]$ , by Lemma 5.7,  $e(2, x) = 1$ , so  $e(2, x; \varepsilon, \delta) = 1$  for suitable choice of  $\varepsilon, \delta$ . By Lemma 2.9, this also holds for  $x \in [3/5, 4/5]$ .
- For  $x \in [0, 1/5]$ , take  $\mathbf{a}_x = (10)$  and  $\mathbf{b}_x = (11)$ . Then  $x(10), x(11) \in K_0$ . By Lemma 5.7,  $(10, 11)$  and  $(11, 10)$  are the only elements in  $E(2, x)$ . So by Proposition 2.2 (3) and Lemma 2.9, the condition (ii) in Lemma 2.12 is satisfied.
- For  $x \in (2/5, 1/2]$ , let  $\mathbf{a}_x = (01)$ ,  $\mathbf{b}_x = (10)$  and  $\mathbf{c}_x = (11)$ . Then  $x(01), x(10) \in K_0$  and  $x(11) \in K_1$ ; and by Lemma 5.7,  $(01, 10)$ ,  $(01, 11)$ ,  $(10, 01)$  and  $(11, 01)$  are the only non-trivial pairs in  $E(2, x)$ . So by Proposition 2.2 (3) and Lemma 2.9, the condition (iii) in Lemma 2.12 is satisfied.

Thus  $\sigma(2) < 1.61$ .  $\square$

### 5.3. Step 3. When $\gamma^2 > \sqrt{2}/4$ .

**Proposition 5.8.** *Assume  $\gamma \leq 0.64$ . Then  $\sigma(2) \leq \sqrt{2}$ .*

An immediate corollary of this proposition and Corollary 5.5 is the following:

**Corollary 5.9.** *If  $\gamma^2 > \sqrt{2}/4$ , then either  $\sigma(1) < 2\gamma$  or  $\sigma(2) < (2\gamma)^2$ .*

The proof of Proposition 5.8 relies on the following estimates.

**Lemma 5.10.** *If  $\gamma \leq 0.64$ , then for any  $x \in [0, 1/2]$ ,  $(01) \not\sim_x (11)$ .*

*Proof.* The case  $0 \leq x \leq 2/5$  was treated in Lemma 5.6. Here we consider the case  $x \in [2/5, 1/2]$ . Let  $\mathbf{u} = \{u_n\}_{n=1}^\infty$  and  $\mathbf{v} = \{v_n\}_{n=1}^\infty$  be such that  $(u_1 u_2) = (01)$  and  $(v_1 v_2) = (11)$ . Let  $F(x) = -(2\pi)^{-1}(S(x, \mathbf{u}) - S(x, \mathbf{v}))$  and let  $P_n, Q_n$  be defined as above. Then

$$P_1 = 2 \sin(\pi x) \geq 2 \sin \frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{2} > 1.902,$$

and

$$P_2 = \cos \frac{\pi x}{2} - \sin \frac{\pi x}{2} > 0.$$

**Case 1.**  $u_3 = 0$  or  $v_3 = 1$ . Then

$$\begin{aligned} P_3 &= (-1)^{u_3} \cos \frac{\pi x}{4} - (-1)^{v_3} \cos \frac{\pi(1+x)}{4} \geq \cos \frac{\pi(1+x)}{4} - \cos \frac{\pi x}{4} \\ &\geq -2 \sin \frac{\pi}{8} \sin \frac{\pi(1+2x)}{8} > -\sqrt{2} \sin \frac{\pi}{8}, \end{aligned}$$

so

$$\begin{aligned} F(x) &\geq P_1 + \gamma P_2 + \gamma^2 P_3 - \frac{2\gamma^3}{1-\gamma} \\ &> 1.902 - \sqrt{2} \cdot 0.64^2 \sin \frac{\pi}{8} - \frac{2 \cdot 0.64^3}{1-0.64} > 0. \end{aligned}$$

**Case 2.**  $u_3 = 1$  and  $v_3 = 0$ . Then

$$\begin{aligned} P_3 &= -\cos \frac{\pi(1+x)}{4} - \cos \frac{\pi x}{4} = -2 \cos \frac{\pi}{8} \cos \frac{\pi(1+2x)}{8} \\ &> -2 \cos^2 \frac{\pi}{8} = -1 - \frac{\sqrt{2}}{2} > -1.708. \end{aligned}$$

*Subcase 2.1.*  $i_4 = 1$  and  $j_4 = 0$ . Then

$$P_4 = -\sin \frac{\pi(2-x)}{8} - \sin \frac{\pi(x+3)}{8} = -2 \sin \frac{5\pi}{16} \cos \frac{\pi(1+2x)}{16} > -2 \sin \frac{5\pi}{16} > -1.663.$$

and

$$P_5 \geq -\sin \frac{\pi(2-x)}{16} - \sin \frac{\pi(x+3)}{16} = -2 \sin \frac{5\pi}{32} \cos \frac{\pi(1+2x)}{32} > -2 \sin \frac{5\pi}{32} > -0.942.$$

Thus

$$\begin{aligned} F(x) &\geq P_1 + \gamma P_2 + \gamma^2 P_3 + \gamma^3 P_4 + \gamma^4 P_5 - \frac{2\gamma^5}{1-\gamma} \\ &\geq 1.902 - 0.64^2 \cdot 1.708 - 0.64^3 \cdot 1.663 - 0.942 \cdot 0.64^4 - \frac{2 \cdot 0.64^5}{1-0.64} > 0.011. \end{aligned}$$

*Subcase 2.2.* Either  $i_4 = 0$  or  $j_4 = 1$ . Then

$$P_4 \geq \sin \frac{\pi(2-x)}{8} - \sin \frac{\pi(3+x)}{8} = -2 \sin \frac{\pi(1+2x)}{16} \cos \frac{5\pi}{16} > -2 \sin \frac{\pi}{8} \cos \frac{5\pi}{16} > -0.556.$$

Thus

$$\begin{aligned} F(x) &\geq P_1 + \gamma P_2 + \gamma^2 P_3 + \gamma^3 P_4 - \frac{2\gamma^4}{1-\gamma} \\ &> 1.902 - 0.64^2 \cdot 1.708 - 0.64^3 \cdot 0.556 - \frac{2 \cdot 0.64^4}{1-0.64} > 0.124. \end{aligned}$$

□

Summarizing the results obtained in Lemma 5.7 and Lemma 5.10, we have

**Lemma 5.11.** *Assume  $\gamma \leq 0.64$ . Then*

- (1) *For  $x \in [1/5, 2/5]$ ,  $e(2, x) = 1$ ;*
- (2) *For  $x \in [0, 1/5)$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(10, 11)$  and  $(11, 10)$ ;*
- (3) *For  $x \in (2/5, 1/2]$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(01, 10)$  and  $(10, 01)$ .*

*Proof.* This follows immediately from Lemma 5.7 and Lemma 5.10. □

*Proof of Proposition 5.8.* Let  $K = [1/5, 2/5] \cup [3/5, 4/5]$ . Then the conditions in Lemma 2.10 are satisfied with  $q = 2$  and suitable choice of  $\varepsilon, \delta$ . Indeed,

- By Lemma 5.11, for each  $x \in [1/5, 2/5]$ , we have  $e(2, x) = 1$ . So by Lemma 2.9 and Proposition 2.2 (3), condition (i) of Lemma 2.10 is satisfied with suitable choices of  $(\varepsilon, \delta)$ ;
- By Lemma 5.11, for each  $x \in [0, 1/5)$ ,  $(10, 11)$  and  $(11, 10)$  are the only non-trivial pairs in  $E(2, x)$ , and it is easily checked  $x(10), x(11) \in K$ . For  $x \in (2/5, 1/2]$ , by Lemma 5.11,  $(01, 10)$  and  $(10, 01)$  are the only non-trivial pairs in  $E(2, x)$  and it is easily checked that  $x(10), x(01) \in$

$K$ . Thus by Lemma 2.9 and Proposition 2.2 (3), condition (ii) of Lemma 2.10 is satisfied.

Thus  $\sigma(2) \leq \sqrt{2}$ .  $\square$

5.4. **When  $\gamma^3 > \sqrt{2}/8$ .**

**Proposition 5.12.** *Assume  $\gamma^2 \leq \sqrt{2}/4$ . Then  $\sigma(3) \leq \sqrt{2}$ .*

An immediate corollary of this proposition and Corollary 5.9 is the following:

**Corollary 5.13.** *If  $\gamma^3 > \sqrt{2}/8$ , then  $\sigma(q) < (2\gamma)^q$  for some  $q \in \{1, 2, 3\}$ .*

**Lemma 5.14.** *Assume  $\gamma^2 \leq \sqrt{2}/4$ . If  $(01u_3) \sim_x (10v_3)$  for some  $x \in [0, 1/2]$  then  $x \in (17/36, 1/2]$ ,  $u_3 = 1$  and  $v_3 = 0$ .*

*Proof.* Note that  $\gamma < 0.6$ . Let  $\mathbf{u} = 01u_3 \cdots$ ,  $\mathbf{v} = 10v_3 \cdots$  and let  $F(x) = -(2\pi)^{-1}(S(x, \mathbf{u}) - S(x, \mathbf{v}))$ ,  $G(x) = F'(x)/\pi$ . We continue to use the notation  $x_n, y_n, P_n$  and  $Q_n$ .

If  $x \leq 0.45$ , then as in the proof of Lemma 5.3,

$$\begin{aligned} G(x) &\geq 2 \cos(\pi x) - \frac{\gamma}{\sqrt{2}} \cos\left(\frac{\pi x}{2} + \frac{\pi}{4}\right) - \frac{\gamma^2}{2-\gamma} \\ &\geq 2 \cos(0.45\pi) - 0.6 \cos(0.475\pi)/\sqrt{2} - 0.36/1.4 > 0.02. \end{aligned}$$

Assume  $(u_3, v_3) \neq (1, 0)$ . We shall show that for each  $x \in [9/20, 1/2]$ ,  $F(x) > 0$ . Indeed, in this case,  $P_1 = 2 \sin(\pi x)$ ,  $P_2 = -\sin \frac{\pi x}{2} - \cos \frac{\pi x}{2}$  and

$$\begin{aligned} P_3 &\geq \cos \frac{\pi(1-x)}{4} - \cos \frac{\pi x}{4} &= -2 \sin \frac{\pi}{8} \sin \frac{\pi(1-2x)}{8} \\ &\geq -2 \sin \frac{\pi}{8} \sin \frac{\pi}{40} &\geq -\frac{\pi^2}{160}. \end{aligned}$$

Thus

$$\begin{aligned} F(x) &\geq P_1 + \gamma P_2 + \gamma^2 P_3 - \frac{2\gamma^3}{1-\gamma} \\ &\geq 2 \sin(\pi x) - \gamma \left( \cos \frac{\pi x}{2} + \sin \frac{\pi x}{2} \right) - \frac{\gamma^2 \pi^2}{160} - \frac{2\gamma^3}{1-\gamma} \\ &\geq 2 \sin(\pi x) - 0.6 \left( \cos \frac{\pi x}{2} + \sin \frac{\pi x}{2} \right) - 0.023 - 1.08. \end{aligned}$$

For  $x \in [9/20, 1/2]$ , this give us

$$F(x) > 2 \sin \frac{9\pi}{20} - 0.6\sqrt{2} - 1.103 > 0.013.$$

Assume now  $(u_3, v_3) = (1, 0)$ . We shall show that  $G(x) > 0$  for all  $x \in [9/20, 17/36]$ . Indeed, in this case,

$$Q_3 = \cos \frac{\pi(2-x)}{4} - \cos \frac{\pi(1+x)}{4} = -2 \sin \frac{3\pi}{8} \sin \frac{\pi(1-2x)}{8} > -\frac{\pi}{40}.$$

Thus

$$\begin{aligned}
G(x) &= Q_1 + \frac{\gamma}{2}Q_2 + \frac{\gamma^2}{4}Q_3 - \frac{\gamma^3}{2-\gamma} \\
&\geq 2\cos(\pi x) - \frac{\gamma}{\sqrt{2}}\cos\left(\frac{\pi x}{2} + \frac{\pi}{4}\right) - \frac{\gamma^2}{4}\frac{\pi}{40} - \frac{\gamma^3}{4-2\gamma} \\
&\geq 2\cos\frac{17\pi}{36} - \frac{0.6}{\sqrt{2}}\cos\frac{35\pi}{72} - 0.01 - 0.08 \\
&> 0.174 - 0.02 - 0.01 - 0.08 > 0.
\end{aligned}$$

□

**Lemma 5.15.** *Assume  $\gamma < 0.6$ . If  $x \in [0, 1/2]$  and  $\mathbf{u}, \mathbf{v}$  are distinct elements in  $\mathcal{A}^3$  such that  $(\mathbf{u}, \mathbf{v}) \in E(3, x)$ , then either*

$$x \in [0, 1/9] \text{ and } (\mathbf{u}, \mathbf{v}) \in \{(101, 110), (110, 101), (010, 011), (011, 010)\};$$

or

$$x \in (17/36, 1/2] \text{ and } (\mathbf{u}, \mathbf{v}) \in \{(011, 100), (100, 011)\}.$$

*Proof.* Lemma 5.14 particularly implies that  $(01) \not\sim_x (10)$  for  $x \in [2/5, 17/36]$ . Together with Lemma 5.11, it follows that

$$(a) \quad 0 \not\sim_x 1 \text{ for } x \in [0, 17/36].$$

By Lemma 2.9,

$$(a') \quad 0 \not\sim_x 1 \text{ for } x \in [19/36, 1].$$

Therefore, for  $x \in [1/18, 1/5]$ ,  $(10) \not\sim_x (11)$ , since  $x(1) \in [19/36, 3/5]$ . Together with Lemma 5.11, we obtain

- (b) For  $x \in [1/18, 17/36]$ ,  $e(2, x) = 1$ ;
- (c) For  $x \in [0, 1/18]$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(10, 11)$  or  $(11, 10)$ ;
- (d) For  $x \in (17/36, 1/2]$ , the only possible non-trivial pairs in  $E(2, x)$  are  $(01, 10)$  or  $(10, 01)$ .

Consider  $x \in [1/9, 17/36]$ . If  $(u_1 u_2 u_3) \sim_x (v_1 v_2 v_3)$  then by (b),  $u_1 u_2 = v_1 v_2$ . Note that for any  $u_1 u_2$ ,  $x(u_1 u_2) \notin [17/36, 19/36]$ . Thus by (a) (and (a')), we have  $u_3 = v_3$ . This proves that  $e(3, x) = 1$ .

Consider  $x \in [0, 1/9]$  and  $(u_1 u_2 u_3) \sim_x (v_1 v_2 v_3)$ . Then by (a),  $u_1 = v_1$ . If  $u_1 = 0$  then  $x(u_1) \in [0, 1/18]$ , and  $(u_2 u_3) \sim_{x(u_1)} (v_2 v_3)$ , so by (c),  $(u_2 u_3, v_2 v_3) \in \{(10, 11), (11, 10)\}$ . If  $u_1 = 1$ , then  $x(u_1) \in [1/2, 10/18]$ , and  $(u_2 u_3) \sim_{x(u_1)} (v_2 v_3)$ , so by Lemma 5.11 (3),  $(u_2 u_3, v_2 v_3) \in \{(01, 10), (10, 01)\}$ .

Consider  $x \in [17/36, 1/2]$ . Then by (d) and Lemma 5.14, the only possible non-trivial pairs in  $E(3, x)$  are  $(011, 100)$  and  $(100, 011)$ . □

*Proof of Proposition 5.12.* Let  $K = [1/9, 17/36] \cup [36/19, 8/9]$ . The conditions in Lemma 2.10 are satisfied with  $q = 3$  and suitable choices of  $(\varepsilon, \delta)$ . Indeed, putting  $L_1 = (17/36, 19/36)$ ,  $L_2 = [0, 1/9] \cup (8/9, 1)$ . by Lemma 5.15, Lemma 2.9 and Proposition 2.2 (3), there exist  $\varepsilon > 0$  and  $\delta > 0$  such that

- $e(3, x; \varepsilon, \delta) = 1$  for all  $x \in K$ .
- For  $x \in L_1$ ,  $e(3, x; \varepsilon, \delta) \leq 2$  and the only non-trivial elements of  $\mathcal{A}^3$  which appears in a non-trivial pair of  $E(3, x)$  are  $(011)$  and  $(100)$  for which we have  $x(011), x(100) \in K$ .

- For  $x \in [0, 1/9)$ ,  $e(3, x; \varepsilon, \delta) \leq 2$  and the only elements of  $\mathcal{A}^3$  which appear in non-trivial pairs of  $E(3, x)$  are (010), (011), (101) and (110) for which  $x(010), x(011), x(101), x(110) \in K$ . By Lemma 2.9, similar properties hold for  $x \in (8/9, 1)$ .

By Lemma 2.10, we have  $\sigma(3) \leq \sqrt{2}$ .  $\square$

### 5.5. Last Step: When $\gamma^3 \leq \sqrt{2}/8$ .

**Proposition 5.16.** *Assume  $\gamma^3 \leq \sqrt{2}/8$ . Then  $e(1) = 1$ .*

*Proof.* It suffices to show that

$$(5.5) \quad (011) \not\prec_x (100) \text{ for } x \in [17/36, 1/2].$$

Indeed, by Lemma 5.15, it follows that  $0 \not\prec_x 1$  for all  $x \in [0, 1/2]$ . By Lemma 2.9,  $0 \not\prec_x 1$  also holds for  $x \in [1/2, 1]$ .

To prove (5.5), let  $\mathbf{u} = (011 \cdots)$ ,  $\mathbf{v} = (100 \cdots)$  and  $F(x) = -(2\pi)^{-1}(S(x, \mathbf{u}) - S(x, \mathbf{v}))$ . We shall prove that  $F(x) > 0$  holds for each  $x \in [17/36, 1/2]$ . We shall continue to use the notation  $x_n, y_n, P_n$  and  $Q_n$  as above. Then  $P_1 = 2 \sin(\pi x)$ ,  $P_2 = -\sin \frac{\pi x}{2} - \cos \frac{\pi x}{2}$  and

$$P_3 = -\sin \frac{\pi(2-x)}{4} - \sin \frac{\pi(1+x)}{4} = -2 \sin \frac{3\pi}{8} \cos \frac{\pi(1-2x)}{8}.$$

Put

$$g(x) = P_1 + \gamma P_2 + \gamma^2 P_3.$$

Using  $\gamma \leq 0.562$ , it is easy to check that  $g'' < 0$  on  $[9/20, 1/2]$ . By calculation,

$$g(9/20) > 0.602 \text{ and } g(1/2) > 0.624.$$

It follows that

$$g(x) > 0.602 \text{ for any } x \in [17/36, 1/2].$$

**Case 1.**  $i_4 = 0$  or  $j_4 = 1$ . Then

$$P_4 \geq -\left(\sin \frac{\pi(2-x)}{8} - \sin \frac{\pi(1+x)}{8}\right) \geq -2 \sin \frac{\pi(1-2x)}{8} \geq -2 \sin \frac{\pi}{144} > -0.05.$$

Then

$$\begin{aligned} F(x) &\geq g(x) + \gamma^3 P_4 - \frac{2\gamma^4}{1-\gamma} \\ &> 0.602 - 0.05\gamma^3 - 2\gamma^4/(1-\gamma) > 0. \end{aligned}$$

**Case 2.**  $i_4 = 1$  and  $j_4 = 0$ . Then

$$P_4 = -\left(\sin \frac{\pi(2-x)}{8} + \sin \frac{\pi(1+x)}{8}\right) \geq -2 \sin \frac{3\pi}{16} > -1.112$$

and

$$P_5 \geq -\left(\sin \frac{\pi(2-x)}{16} + \sin \frac{\pi(1+x)}{16}\right) \geq -2 \sin \frac{3\pi}{32} > -0.581.$$

Thus

$$F(x) \geq g(x) + \gamma^3 P_4 + \gamma^4 P_5 - \frac{2\gamma^5}{1-\gamma} > 0.602 - 1.112\gamma^3 - 0.581\gamma^4 - \frac{2\gamma^5}{1-\gamma} > 0.09.$$

$\square$

## 6. APPENDIX: A PROOF OF LEDRAPPIER'S THEOREM

This appendix is devoted to a proof of Ledrappier's theorem under a further assumption that  $\phi'$  is continuous (for simplicity). The proof is motivated by the original proof given in [10] and also the recent paper [9].

Let  $b \geq 2$  be an integer and  $\lambda \in (1/b, 1)$  and let  $f(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x)$ . We use  $z$  to denote a point in  $\mathbb{R}^2$  and  $B(z, r)$  denote the open ball in  $\mathbb{R}^2$  centered at  $z$  and of radius  $r$ . We assume that  $\dim(m_x) = 1$  holds for Lebesgue a.e.  $x \in [0, 1)$ , which means for Lebesgue a.e.  $x \in [0, 1)$  and  $\mathbb{P}$ -a.e.  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$ ,

$$\lim_{r \rightarrow 0} \frac{\log \mathbb{P}(\{\mathbf{v} : |S(x, \mathbf{u}) - S(x, \mathbf{v})| \leq r\})}{\log r} = 1.$$

Let  $\mu$  be the pushforward of the Lebesgue measure on  $[0, 1)$  under the map  $x \mapsto (x, f(x))$ . Let

$$\underline{d}(\mu, z) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(z, r))}{\log r}$$

and

$$\underline{D} = \text{essinf } \underline{d}(\mu, z).$$

We shall prove that

$$\underline{D} \geq D = 2 + \frac{\log \lambda}{\log b}.$$

This is enough to conclude that the Hausdorff dimension of the graph of  $f$  is  $D$ . Indeed, by the mass distribution principle, it implies that the Hausdorff dimension is at least  $D$ . On the other hand, it is easy to check that  $f$  is a  $C^{2-D}$  function which implies that the Hausdorff dimension is at most  $D$  (see for example Theorem 8.1 of [5]).

**6.1. Telescope.** Define  $\Phi : [0, 1) \times \mathbb{R} \rightarrow [0, 1) \times \mathbb{R}$  as

$$\Phi(x, y) = (bx \bmod 1, (y - \phi(x))/\lambda).$$

Define  $G : [0, 1) \times \mathbb{R} \times \mathcal{A}^{\mathbb{Z}^+} \rightarrow [0, 1) \times \mathcal{A}^{\mathbb{Z}^+}$  as

$$G(x, y, \mathbf{u}) = (\Phi(x, y), u_0 \mathbf{u}), \text{ if } bx \in [u_0, u_0 + 1).$$

The graph of  $f$  is an invariant repeller of the expanding map  $\Phi$ . We shall use neighborhoods bounded by unstable manifolds. For each  $z_0 = (x_0, y_0) \in \mathbb{R}^2$  and  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$ , let  $\ell_{z_0, \mathbf{u}}(x)$  denote the unique solution of the initial value problem:

$$y' = -\gamma S(x, \mathbf{u}), y(x_0) = y_0.$$

These curves are strong unstable manifolds of  $\Phi$  and they satisfy the following property: for  $z = (x, y)$ ,  $z_0 = (x_0, y_0) \in [u_0/b, (u_0 + 1)/b) \times \mathbb{R}$ ,  $u_0 \in \mathcal{A}$ ,

$$\Phi(x, \ell_{z_0, \mathbf{u}}(x)) = (bx - u_0, \ell_{\Phi(z_0), u_0 \mathbf{u}}(bx - u_0)).$$

For  $z_0 = (x_0, y_0) \in [0, 1) \times \mathbb{R}$ ,  $\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}$  and  $\delta_1, \delta_2 > 0$ , let

$$Q(z_0, \mathbf{u}, \delta_1, \delta_2) = \{(x, y) : x \in [0, 1), |x - x_0| \leq \delta_1, |y - \ell_{z_0, \mathbf{u}}(x)| \leq \delta_2\}.$$

The following observation was taken from [9].

**Lemma 6.1** (Telescope). *Let  $\{(z_i, \mathbf{u}_i)\}_{i=0}^n$  be a  $G$ -orbit and let  $x_i$  denote the first coordinate of  $z_i$ . For any  $\delta_1, \delta_2 > 0$ , if  $\delta_1 \leq x_n < 1 - \delta_1$ , then*

$$\mu(Q(z_0, \mathbf{u}, \delta_1 b^{-n}, \delta_2 \lambda^n)) = b^{-n} \mu(Q(z_n, \mathbf{u}_n, \delta_1, \delta_2)).$$

*Proof.* Let  $J_i = [x_i - \delta_1 b^{i-n}, x_i + \delta_1 b^{i-n}]$ ,  $Q_i = Q(z_i, \mathbf{u}_i, \delta_1 b^{i-n}, \delta_2 \lambda^{n-i})$  and let  $E_i = \{x \in J_i : (x, f(x)) \in Q_i\}$ . Then  $\mu(Q_i) = |E_i|$ . Under the assumption  $\delta_1 \leq x_n < 1 - \delta_1$ ,  $Q_0$  is mapped onto  $Q_n$  diffeomorphically under  $\Phi^n$ . Thus  $J_0$  is mapped onto  $J_n$  and  $E_0$  is mapped onto  $E_n$  diffeomorphically under the linear map  $x \mapsto b^n x$ . Thus  $|E_0| = b^{-n}|E_n|$ .  $\square$

**6.2. A version of Marstrand's estimate.** Fix a constant  $t \in (1/(1+\alpha), 1)$ .

**Proposition 6.2.** For  $\mu \times \mathbb{P}$ -a.e.  $(z_0, \mathbf{u})$ ,

$$(6.1) \quad \liminf_{r \rightarrow 0} \frac{\log \mu(Q(z_0, \mathbf{u}, r^t, r))}{\log r} \geq 1 + t(\underline{D} - 1).$$

*Proof.* It suffices to prove that for each  $\xi > 0$  and  $\eta > 0$ , there is a subset  $\Sigma$  of  $[0, 1) \times \mathbb{R} \times \mathcal{A}^{\mathbb{Z}^+}$  with  $(\mu \times \mathbb{P})(\Sigma) > 1 - \eta$  such that

$$(6.2) \quad \liminf_{r \rightarrow 0} \frac{\log \mu(Q(z_0, \mathbf{u}, r^t, r))}{\log r} \geq 1 + t(\underline{D} - 1) - 3\xi$$

holds for all  $(z_0, \mathbf{u}) \in \Sigma$ . By Egoroff's theorem, we can choose  $\Sigma$  with  $(\mu \times \mathbb{P})(\Sigma) > 1 - \eta$  for which there is  $r_0 > 0$  such that for each  $(z_0, \mathbf{u}) \in \Sigma$ ,

(S1)  $\mathbb{P}(\{\mathbf{v} : |S(x_0, \mathbf{u}) - S(x_0, \mathbf{v})| \leq r\}) \leq r^{1-\xi}$  for each  $0 < r \leq r_0$ , where  $x_0$  is the first coordinate of  $z_0$ ;

(S2)  $\mu(B(z_0, r)) \leq r^{\underline{D}-\xi}$  for each  $0 < r \leq r_0$ .

In the following we shall prove that for  $r > 0$  small enough,

$$(6.3) \quad \int_{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma} \mu(Q(z_0, \mathbf{u}, r^t, r)) d\mathbb{P} \leq r^{1+t(\underline{D}-1)-2\xi},$$

holds for every  $z_0 \in [0, 1) \times \mathbb{R}$ . This is enough to conclude the proof. Indeed, let  $\tau \in (0, 1)$  be an arbitrary constant. Then by (6.3), there is  $N$  such that for  $n > N$ ,

$$\mathbb{P}\left(\left\{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma, \mu(Q(z_0, \mathbf{u}, \tau^{nt}, \tau^n)) > (\tau^n)^{1+t(\underline{D}-1)-3\xi}\right\}\right) \leq \tau^{n\xi}$$

holds for every  $z_0 \in [0, 1) \times \mathbb{R}$ . By Fubini's theorem, this implies that

$$\mu \times \mathbb{P}\left(\left\{(z_0, \mathbf{u}) \in \Sigma : \mu(Q(z_0, \mathbf{u}, \tau^{nt}, \tau^n)) > (\tau^n)^{1+t(\underline{D}-1)-3\xi}\right\}\right) \leq \tau^{n\xi}.$$

By Borel-Cantelli, it follows that for almost every  $(z_0, \mathbf{u}) \in \Sigma$ ,  $\mu(Q(z_0, \mathbf{u}, \tau^{nt}, \tau^n)) \leq (\tau^n)^{1+t(\underline{D}-1)-3\xi}$  holds for all  $n$  large enough. The inequality (6.2) follows.

Let us now prove (6.3). We first prove

**Claim.** Provided that  $r > 0$  is small enough, for every  $z_0, z \in [0, 1) \times \mathbb{R}$ , we have

$$(6.4) \quad \mathbb{P}(\{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma, z \in Q(z_0, \mathbf{u}, r^t, r)\}) \leq C_1 \left(\frac{r}{|z - z_0|}\right)^{1-2\xi},$$

where  $C_1 > 0$  is a constant.

To prove this claim, let  $z = (x, y)$ ,  $z_0 = (x_0, y_0)$  and  $h(x) = \ell_{z_0, \mathbf{u}}(x)$ . Then  $h(x)$  is  $C^{1+\alpha}$  with uniformly bounded norm. So

$$\begin{aligned} |y - y_0 + \gamma S(x_0, \mathbf{u})(x - x_0)| &\leq |y - h(x)| + |h(x) - h(x_0) - h'(x_0)(x - x_0)| \\ &\leq r + \left| \int_{x_0}^x (h'(s) - h'(x_0)) ds \right| \leq r + Cr^{t(1+\alpha)} < 2r, \end{aligned}$$

provided that  $r$  is small enough. Thus

$$\{S(x_0, \mathbf{u}) : (z_0, \mathbf{u}) \in \Sigma, z \in Q(z_0, \mathbf{u}, r^t, r)\}$$

is contained in an interval of length  $2r/(\gamma|x - x_0|)$ . Since

$$|z - z_0| \leq |x - x_0| + |y - y_0| \leq (1 + \gamma|S(x_0, \mathbf{u})|)|x - x_0| + 2r,$$

and  $|S(x, \mathbf{u})|$  is uniformly bounded, the inequality (6.4) follows from the property (S1). Note that if  $2r/(\gamma|x - x_0|) > r_0$ , then  $r/|z - z_0|$  is bounded away from zero, so (6.4) holds for sufficiently large  $C_1$ , since the left hand side of this inequality does not exceed one.

We continue the proof of (6.3). Note that there is a constant  $C_2 > 0$  such that for every  $r > 0$  and any  $z_0 \in [0, 1) \times \mathbb{R}$ ,

$$\bigcup_{\mathbf{u} \in \mathcal{A}^{\mathbb{Z}^+}} Q(z_0, \mathbf{u}, r^t, r) \subset B(z_0, C_2 r^t).$$

Of course we may assume there is  $\mathbf{u}$  such that  $(z_0, \mathbf{u}) \in \Sigma$ . Thus for  $R > 0$  small enough, we may apply (S2) and obtain

$$\begin{aligned} \int_{B(z_0, R)} \frac{d\mu(z)}{|z - z_0|^{1-2\xi}} &= \sum_{n=0}^{\infty} \int_{e^{-n-1}R \leq |z - z_0| < e^{-n}R} \frac{d\mu(z)}{|z - z_0|^{1-2\xi}} \\ &\leq \sum_{n=0}^{\infty} \frac{\mu(B(z_0, e^{-n}R))}{(e^{-n-1}R)^{1-2\xi}} \leq \sum_{n=0}^{\infty} \frac{(e^{-n}R)^{\underline{D}-\xi}}{(e^{-n-1}R)^{1-2\xi}} \\ &= C(\xi)R^{\underline{D}-1+\xi}, \end{aligned}$$

where  $C(\xi)$  is a constant depending on  $\xi$  and  $\underline{D}$ . By Fubini's theorem,

$$\begin{aligned} &\int_{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma} \mu(Q(z_0, \mathbf{u}, r^t, r)) d\mathbb{P}(\mathbf{u}) \\ &= \int_{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma} \int_{\mathbb{R}^2} 1_{Q(z_0, \mathbf{u}, r^t, r)}(z) d\mu(z) d\mathbb{P}(\mathbf{u}) \\ &= \int_{B(z_0, C_2 r^t)} \mathbb{P}(\{\mathbf{u} : (z_0, \mathbf{u}) \in \Sigma, z \in Q(z_0, \mathbf{u}, r^t, r)\}) d\mu(z) \\ &\leq C_1 r^{1-2\xi} \int_{B(z_0, C_2 r^t)} \frac{d\mu(z)}{|z - z_0|^{1-2\xi}} \\ &\leq C' r^{1+t(\underline{D}-1)-(2-t)\xi} < r^{1+t(\underline{D}-1)-2\xi}, \end{aligned}$$

provided that  $r$  is small enough.  $\square$

We are ready to complete the proof of Ledrappier's theorem. For any  $\xi > 0$ ,  $\eta > 0$ , by Proposition 6.2 and Egoff's theorem, we can pick up a subset  $\Sigma$  of  $\mathbb{R}^2 \times \mathcal{A}^{\mathbb{Z}^+}$  and a constant  $r_* > 0$  such that  $(\mu \times \mathbb{P})(\Sigma) > 1 - 3\eta$  and such that for each  $(z, \mathbf{u}) \in \Sigma$ ,

$$\mu(Q(z, \mathbf{u}, r^t, r)) \leq r^{1+t(\underline{D}-1)-\xi} \text{ for each } 0 < r < r_*.$$

We may further assume that  $\Sigma \subset [\eta, 1 - \eta] \times \mathbb{R} \times \mathcal{A}^{\mathbb{Z}^+}$ .

Note that  $\mu \times \mathbb{P}$  is an ergodic invariant measure for the map  $G$ . By Birkhoff's Ergodic Theorem, for almost every  $(z_0, \mathbf{u}_0)$ , there is an increasing sequence



$\{n_k\}_{k=1}^\infty$  of positive integers such that  $G^{n_k}(z_0, \mathbf{u}_0) \in \Sigma$  and

$$(6.5) \quad \liminf_{k \rightarrow \infty} n_k/n_{k+1} > 1 - 3\eta.$$

For each  $n = 1, 2, \dots$ , put  $\delta_n = \gamma^{nt/(1-t)}b^{-n}$ ,  $r_n = \gamma^{n/(1-t)}$ , so that

$$r_n = \delta_n \lambda^{-n}, \text{ and } r_n^t = \delta_n b^n.$$

Let us prove that for  $k$  sufficiently large,

$$(6.6) \quad \frac{\log \mu(Q(z_0, \mathbf{u}_0, \delta_{n_k}, \delta_{n_k}))}{\log \delta_{n_k}} \geq \underline{D} + (D - \underline{D})A_1 - A_2\xi,$$

where  $A_1, A_2$  are positive constants depending only on  $\lambda$  and  $b$ .

Indeed, by Lemma 6.1, for  $k$  large enough,

$$\mu(Q(z_0, \mathbf{u}_0, \delta_{n_k}, \delta_{n_k})) = \frac{\mu(Q(G^{n_k}(z_0, \mathbf{u}_0), r_{n_k}^t, r_{n_k}))}{b^{n_k}} \leq \frac{r_{n_k}^{1+t(\underline{D}-1)-\xi}}{b^{n_k}}.$$

Using definition of  $r_n$  and  $\delta_n$ , this gives us

$$\mu(Q(z_0, \mathbf{u}_0, \delta_{n_k}, \delta_{n_k})) \leq \delta_{n_k}^{\underline{D}} \times (b^{-n_k})^{D-\underline{D}} r_{n_k}^{-\xi},$$

Thus (6.6) holds with  $A_1 = \log b / (\log b + t \log \gamma^{-1}/(1-t))$  and  $A_2 = \log \gamma / (t \log \gamma + (1-t) \log b^{-1})$ .

By (6.5), for each  $n$  large enough, there is  $k$  such that  $(1-3\eta)n_k < n_{k-1} < n \leq n_k$ . It follows that

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(Q(z_0, \mathbf{u}_0, \delta_n, \delta_n))}{\log \delta_n} \geq (1-3\eta)(\underline{D} + (D - \underline{D})A_1 - A_2\xi).$$

Since  $\ell_{x_0, \mathbf{u}_0}$  is a smooth curve, there exists  $\kappa \in (0, 1)$  such that  $Q(z_0, \mathbf{u}_0, \delta_k, \delta_k)$  contains  $B(z_0, \kappa \delta_k)$  for each  $k$ . Therefore,

$$\underline{d}(\mu, z_0) = \liminf_{n \rightarrow \infty} \frac{\log \mu(Q(z_0, \mathbf{u}_0, \delta_n, \delta_n))}{\log \delta_n} \geq (1-3\eta)(\underline{D} + (D - \underline{D})A_1 - A_2\xi).$$

Since this estimate holds for  $\mu$ -a.e.  $z_0$ , we obtain

$$\underline{D} \geq (1-3\eta)(\underline{D} + A_1(D - \underline{D}) - A_2\xi).$$

As  $\xi, \eta$  can be chosen arbitrarily small, we conclude

$$\underline{D} \geq \underline{D} + A_1(D - \underline{D}),$$

which means  $\underline{D} \geq D$ , as desired.

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